Uphoff

Northern Illinois College of Optometry

Theoretical Optics

Study Outlines

THEORETIC OPTICS.

CHAPTER I.

REFLECTION

I. Plane Mirror.

A. Single Mirror.

1. Properties.

A plane mirror possesses the remarkable property of converting a homocentric bundle of rays into another homocentric bundle of rays. No other optical device is capable of this except under conditions that are impracticable.

"To every homocentric bundle of incident rays reflected at a plane mirror there corresponds also a homocentric bundle of reflected rays."

Because of this, it can be easily shown that:

a. Images formed by a plane mirror are always virtual.

b. Lines joining corresponding points in an object and its image are always bisected at right angles by the mirror.

2. Formation of Images (Geometrical).

a. General.

(1) Take a general case; make no assumptions. Incline the object with respect to the mirror.

(2) Use distinctive marks to show if the image is inverted or erect.

(3) Use unbroken and broken lines, or different colors, to distinguish the "object" and "image" portions of the diagram.

b. Specific

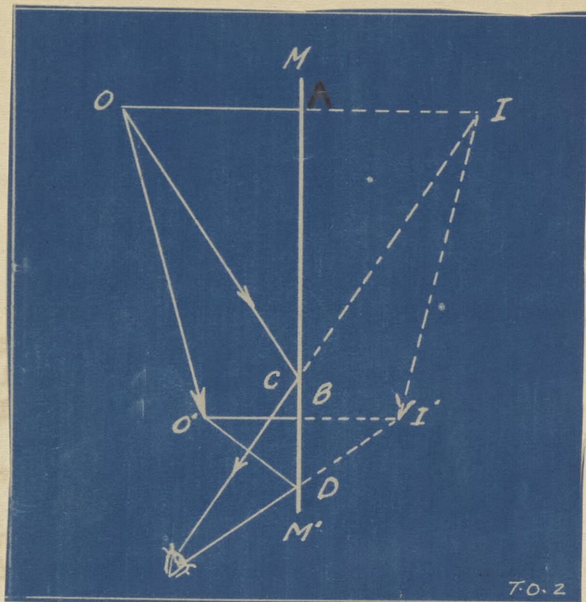
1st Method

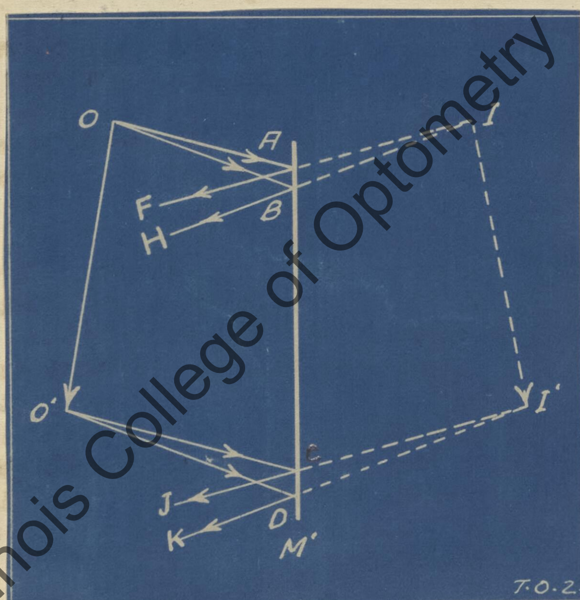
MM' represents a plane mirror and OO' the object.

From the extremities of the object draw two lines

2nd Method

From each extremity of the object, draw a pair of divergent rays; (OA,OB) and (O'C, O'D).


M
O
A
I
C
B
O'
I'
D
M'
T.O.2


O
A
I
F
B
H
O'
I'
C
J
D
K
M'
T.O.2.

OA and O'B perpendicular to MM' and extend them back of MM' so that OA = AI and O'B = BI'.

Now draw two convergent lines OC and O'D from the extremities of the object; and at C and D draw the reflected rays. They will meet after reflection at E.

EC and ED extended beyond MM' should pass through I and I' respectively.

II' represents the image.

At the points of incidence, draw the reflected rays (AF, BH) and (CJ,DK).

Traject these pairs of reflected rays backwards until they meet at I and I' respectively.

II' represents the image.

3. Field of View.

a. Explanation.

The open or visible space commanded by the eye is called the field of view.

The field of view of a plane mirror is enclosed by the outermost rays, which when reflected, will enter the observer's eye. Anything within this space will be seen by the observer, while anything outside will not be seen.

The field of view is limited by the contour of the plane mirror. It is the same as if the observer were looking into the image-space through an aperture which coincided with the place occupied by the mirror.

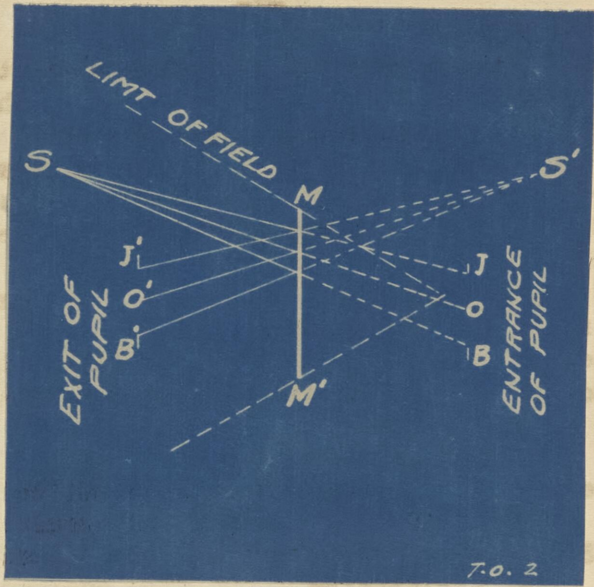
b. Construction.

MM' is a plane mirror and S is a point-source of light within its field of view.

O' is the center of the observer's pupil, while J' and B' are the outside limits of that pupil.

J,O, and B are the images of J',O' and B' respectively.

Join S to the image-points J,O, and B; and where these lines intersect the mirror MM' join the points of intersection to


LIMT OF FIELD
S
M
S'
J'
J
O'
O
B'
B
M'
EXIT OF PUPIL
ENTRANCE OF PUPIL
T.O. 2

their respective object-points J',O', and B'.

These latter lines are divergent; so that if trajected backwards they will meet at a point S'.

S' is the image of S.

c. Definitions.

(1) Field Stop.

"Any agent which limits the extent of the field of vision in an optical system."

(2) Aperture Stop.

"Any agent in an optical system which regulates the angular apertures of the cones of rays entering the observer's eye from each point of a luminous object."

(Notes:

(a) The field stop controls the extent of the field of view.

(b) The aperture stop controls the brightness of the source.)

(3) Entrance Pupil.

"The imaginary opening or virtual stop towards which the incident rays must all be directed in order to be reflected into the observer's pupil, is called the "entrance-pupil" of the optical system."

It is the image of the aperture stop as seen by looking into an instrument in the direction of the light coming from the object.

(4) Exit Pupil.

"The image of the aperture stop as seen by looking into an instrument from the image side."

In the optical system comprising the observer's eye and a plane mirror, the observer's pupil itself is called the "exit-pupil."


d_c
C
LIMIT OF FIELD
B
C
P
A
P'
M'
d_p
d_p
T.O.4

d. Calculations (Reversed Test-Types).

The size of the smallest mirror which will just reflect all of a test-chart depends upon:-

(1) The size of the chart (S_c).
(2) The distance of the chart from the mirror (d_c).
(3) The distance of the observer from the mirror (d_p).

CC' represents the chart and MM' the mirror.

P represents the patient's eye and P' is its image.

The whole chart will just be reflected in the mirror when it just comes within the field of view.

Now the triangles CC'P' and MM'P' are similar.

$$\therefore \frac{MM'}{CC'} = \frac{P'A}{P'B}$$

$$\text{or} \quad \frac{S_m}{S_c} = \frac{d_p}{d_p + d_c}$$

$$\text{and} \quad S_m = \frac{S_c \cdot d_p}{d_p + d_c}$$

$$\boxed{S_m = \frac{d_p}{d_p + d_c} \cdot S_c}$$ 1.

(Note:-
When the chart is just above the head of the patient, then $d_p = d_c$ and the mirror necessary is just one-half the size of the chart.)

II. Inclined Mirrors.

A. Introduction.

When a luminous point lies in the dihedral angle between two inclined, plane mirrors, some of its rays will fall on one mirror and some on the other, while some will be successively reflected from both mirrors before they finally enter the eye of an observer. Consequently there will be several images formed; the number depending upon the angle of inclination of the mirrors.

It is a matter of geometrical proof to show that all these images will lie on the circumference of the circle whose center is the point of intersection of the mirrors and whose radius is the distance from this point to the object-point.

B. Formation of Images.

1. Special Cases.

a. Equal Sectors.

The easiest case to understand is that, in which the 360° of the circle passing through all the images, is an exact multiple of the dihedral angle between the mirrors. The formula for calculating the number of images is then as follows:-

$$\frac{360^\circ}{A^\circ} = n \text{ (including the object).}$$

or $\boxed{\frac{360^\circ}{A^\circ} = n-1}$ (Where A° is the number of degrees in the dihedral angle.) 2.

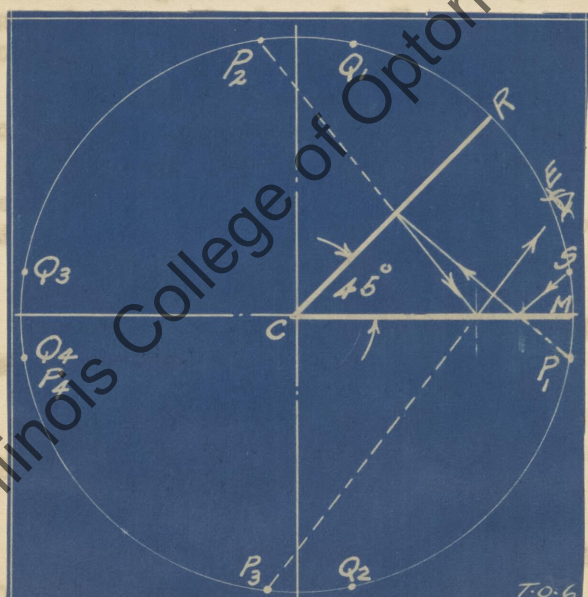
b. Parallel Mirrors.

Theoretically, the number of images is infinite, because $\frac{360}{0} = \infty$.

Practically, the number of images seen depends upon the intensity of the light source.

2. General Case.

Because there are two inclined mirrors it follows that each mirror will have its own


P2
Q1
R
E
S
Q3
45°
M
C
Q4
P4
P1
P3
Q2
T.O.6

series of images. The number in each series depends upon the position of the object; that is, upon the angle that the line joining the object with the point of intersection of the mirrors, makes with each mirror.

Let the images formed by reflection first at CM be known as the P-series; and those formed by reflection first at CR be known as the Q-series.

Then, the total number of images of the P-series, whether it be odd or even, is given by the integer next higher than:-

$$\boxed{\frac{180 - B}{(A + B)}} \quad \text{--- P-series}$$

3

and of the Q-series by the integer next higher than:-

4

$$\boxed{\frac{180 - A}{(A + B)}} \quad \text{--- Q-series.}$$

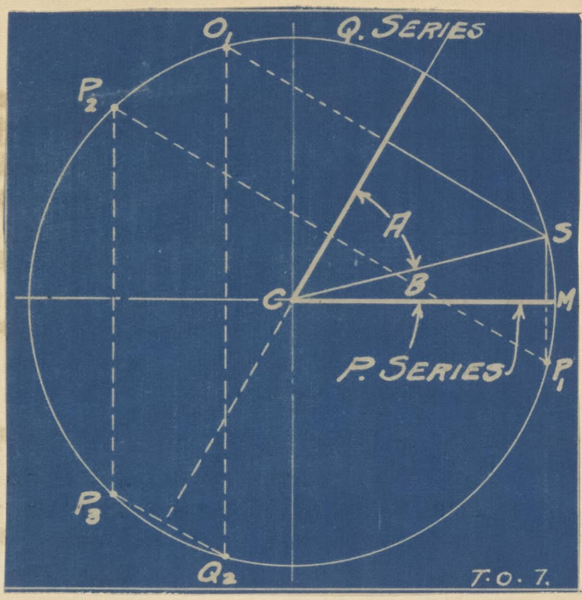
(Note:

The only exception to this rule is when (A + B) is contained in (180-A) or (180-B) an exact number of times. We must then take the actual integer obtained by the division.)

C. Construction of Ray-Paths.

To trace the paths of the rays by which a spectator standing in front of a pair of inclined mirrors sees any image of a luminous point, proceed as follows:-

1. With the point of intersection of the mirrors as center, construct a circle which will pass through the point representing the light source.
2. On this circle mark the position of every image, dividing them into a P-series and a Q-series.


O1
Q. SERIES
P2
A
S
B
C
M
P. SERIES
P1
P3
Q2
T.O. 7.

3. Draw a straight line from a given image-point to the eye; for it is along this line that the light entering the eye arrives.
4. Join the point where this line crosses the mirror in which the image is produced, to the next image-point in the series.
5. Now join the point where the latter line crosses its mirror to the next image-point in the series, and continue until you finally arrive at the light-source.

(Note: If, on joining a point of intersection to the next image-point in the series, the line would not cross its mirror, then that image will not be seen by the eye and it may be omitted from the series.)

D. Successive Reflections.

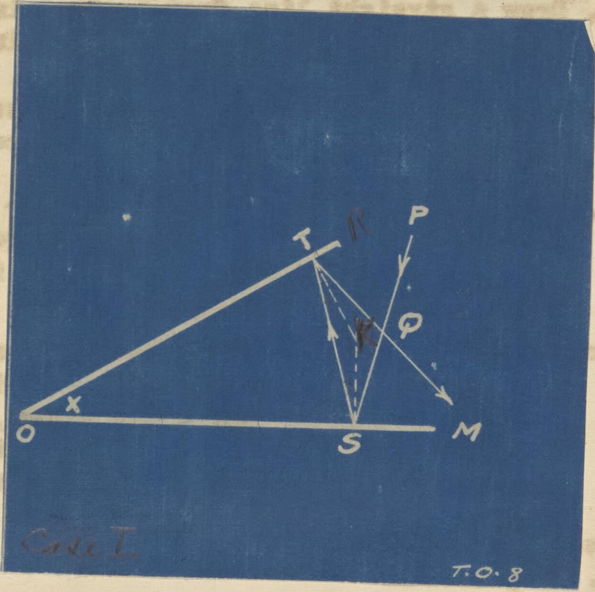
1. Proposition.

   "If a ray lying in a principal section is reflected successively at two plane mirrors, it will be deviated from its original direction by an angle equal to twice the dihedral angle between the mirrors."

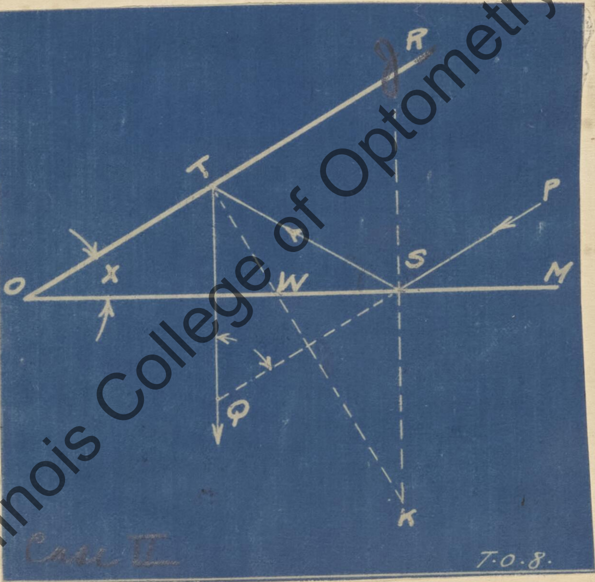
2. Proof.

   There are two possible solutions, as illustrated in the accompanying diagrams.

Diagram 1. Diagram 2.



Case I



Case II

Case 1.

To prove PQT = 2MOR

$$\begin{aligned} PQT &= QTS + QST \\ &= 2KTS + 2KST \\ &= 2(KTS + KST) \\ &= 2(180 - TKS) \end{aligned}$$

But

$TKS + KSO + KTO + MOR = 360^\circ$

and $KTO + KSO = 180^\circ$

$\therefore MOR + TKS = 180^\circ$

and $(180 - TKS) = MOR$

Hence:-

$2(180 - TKS) = 2MOR.$

i.e. $PQT = 2MOR.$

Case 2.

$$\begin{aligned} PST &= STQ + SQT \\ \therefore SQT &= PST - STQ \\ &= 2JST - 2STK \\ &= 2(JST - STK) \\ &= 2SKT \end{aligned}$$

But in the $\triangle$s WOT and WKS

$TWO = SWK$

$WTO = WSK$

$\therefore WOT = SKW = SKT$

Hence:-

$SQT = 2SKT = 2WOT$

i.e. $PQT = 2MOR.$

III. Application of a Plane Mirror.

A. Optometrical.

1. Doubling the testing-space.
2. Retinoscope.

B. Other applications.

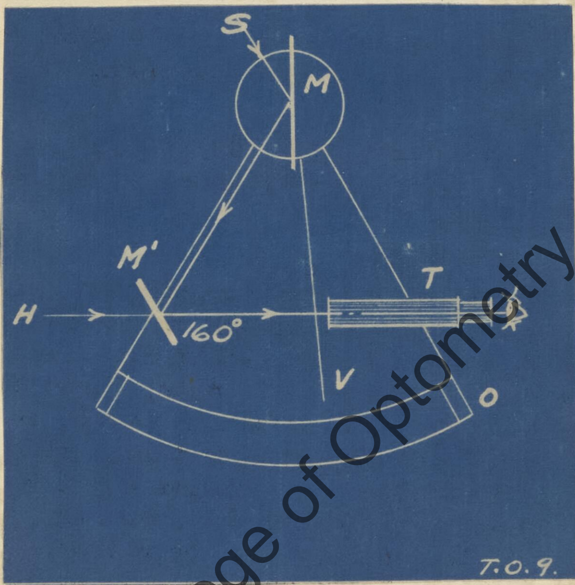
1. Porte Lumière.

This is a plane mirror mounted so as to be capable of rotation about two rectangular axes, whereby it may be readily adjusted to any desired azimuth and reflect a beam of light to any desired place.

Owing to the diurnal movement of the sun, a continuous adjustment of the mirror is necessary, to keep the spot of light in the same place for any length of time. This adjustment is inconvenient. To overcome this inconvenience, a heliostat is used.

2. Heliostat.

This consists of a plane mirror which is continuously revolved by clockwork around an axis parallel to the earth's axis so as to preserve the same relative position with re-


S
M
M'
T
H
160°
V
O
T.O.9.

spect to the sun.

The heliostat is also provided with a plane mirror which can be fixed so as to reflect the rays from the revolving mirror in any required direction.

3. Sextant.

This is an instrument used in measuring the angle subtended, at the eye of the observer, by the line joining two distant objects. By its aid sailors are enabled to measure the altitude of the sun and thus determine the latitude.

The essential parts of a sextant are:-

a. A fixed mirror with the upper half transparent (M_1)
b. A rotating mirror (M_2)
c. A sliding scale attached to the rotating mirror.
d. A telescope.

An object H, (say the horizon), is sighted by the telescope through the unsilvered part of M_1 and the other object S, - (say the sun), is sighted after reflection, first at M_2 and then at M_1. The amount M_2 has to be rotated to bring the second object into view is measured directly by the scales O and V.

Since a reflected ray moves through twice the angle described by the mirror, the readings on the scale must be multiplied by 2. To save the trouble of multiplication, the half-degrees on the scale are numbered as whole degrees.

Because the sextant method is not accurate for angles greater than 120°, it is not necessary to have a graduated arc more than 60° or one sixth part of a circle; whence the name sextant is derived.

4. Kaleidoscope.

This was originally a toy, devised by Sir Edward Brewster (1781-1868). It consists

essentially of two long, narrow strips of mirror glass inclined to each other at an angle of 60° and inclosed in a cylindrical tube. One end of the tube is closed by ground glass on which are loosely disposed a lot of fragments of varicolored glass. At the other end of the tube is a peep-hole.

When the instrument is held towards the light, an observer looking in it will see an exquisitely beautiful and symmetrical pattern formed by the colored objects and their images, the form of which may be endlessly varied by revolving the tube about its axis so that the bits of colored glass assume new positions.

The device has been turned to practical use in making designs for carpets and wall-papers.

IV. Spherical Mirrors.

A. General

1. Tangent.

Instead of representing a spherical mirror by the arc of a circle, it is often more convenient to represent the mirror by a straight line tangent to the arc at the vertex.

2. Definition.

The formulae for spherical mirrors apply to paraxial rays only.

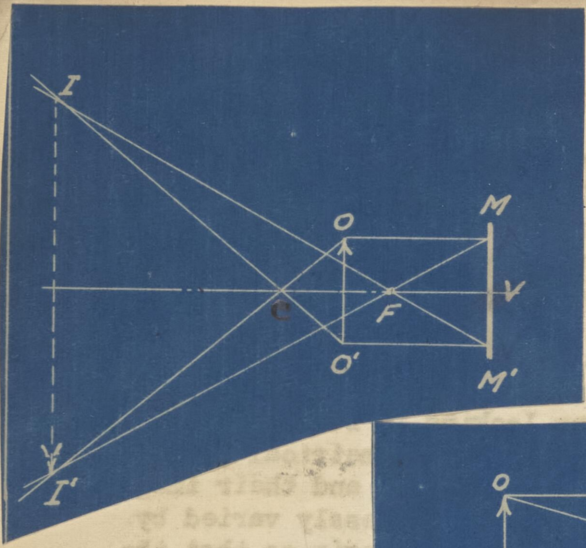
"A paraxial ray is one whose path lies very near the axis of the spherical surface and which therefore meets this surface at a point close to the vertex and at nearly normal incidence."

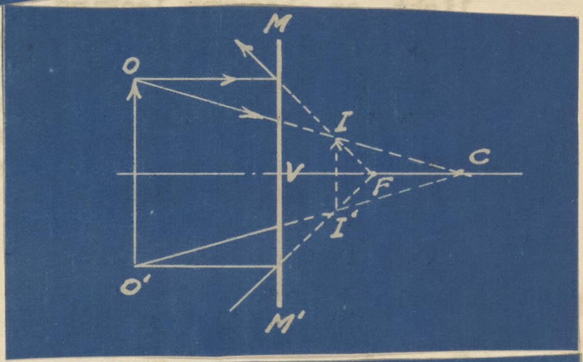
3. Terms.

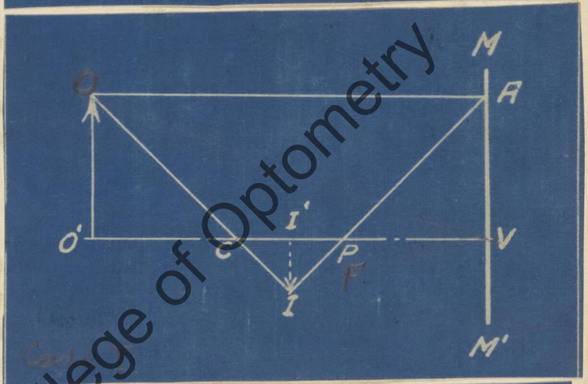
a. Points designated by F_1 and F_2 are principal foci. These letters are also used to designate the principal focal lengths.

b. The distance of the object from the vertex V is represented by f_1 and that of the image by f_2.

c. C marks the center of curvature of the mirror and its distance from the vertex is represented by r.


I
O
M
C
F
V
O'
M'
I'


M
O
I
V
F
C
I'
O'
M'


M
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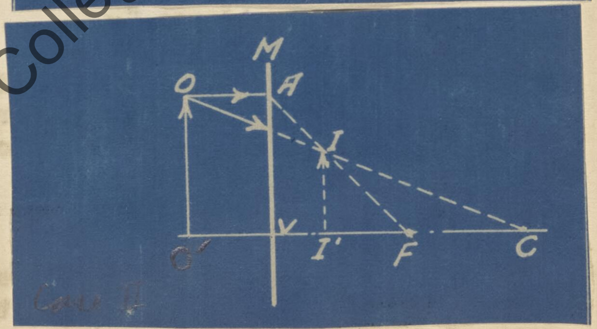
A
I'
O'
C
P
V
F
I
M'


M
O
A
I
V
O'
I'
F
C

B. Graphical Imagery.

Concave and Convex.

1. From each of the two extremities of the object, draw two rays:-
   a. One parallel to the axis and reflected through the principal focus (F)
   b. One through the center of curvature (C) and reflected back along its own path.
2. The point where two rays from the same extremity meet after reflection, will be an image-point of that extremity.

(Notes:
(1) All images formed by a convex mirror are virtual.
(2) Images produced by a concave mirror may be either real or virtual according to the position of the object.)

C. Conjugate Formula.

1. Concave Mirror.

Triangles OCO' and ICI' are similar.

$$\therefore \frac{OO'}{II'} = \frac{O'C}{I'C}$$

Triangles AFV and IFI' are similar.

$$\therefore \frac{AV}{II'} = \frac{VF}{I'F}$$

Since AV = OO'

$$\frac{O'C}{I'C} = \frac{VF}{I'F}$$

$$\frac{f_1 - 2F}{2F - f_2} = \frac{F}{f_2 - F}$$

2. Convex Mirror.

Triangles OCO' and ICI' are similar.

$$\therefore \frac{OO'}{II'} = \frac{O'C}{I'C}$$

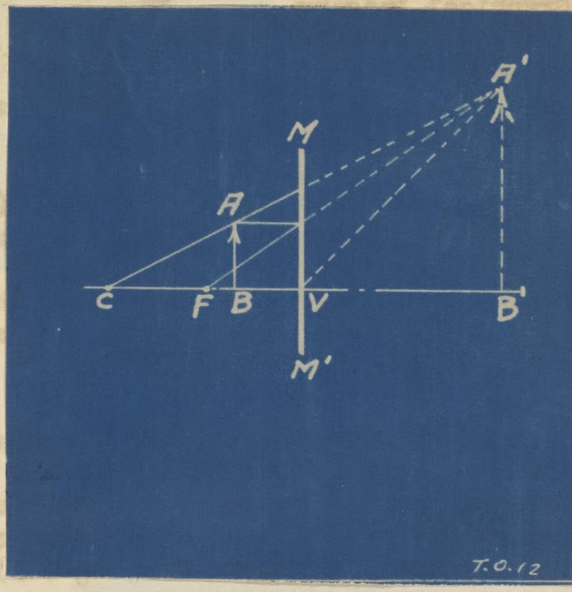
Triangles AVF and IFI' are similar.

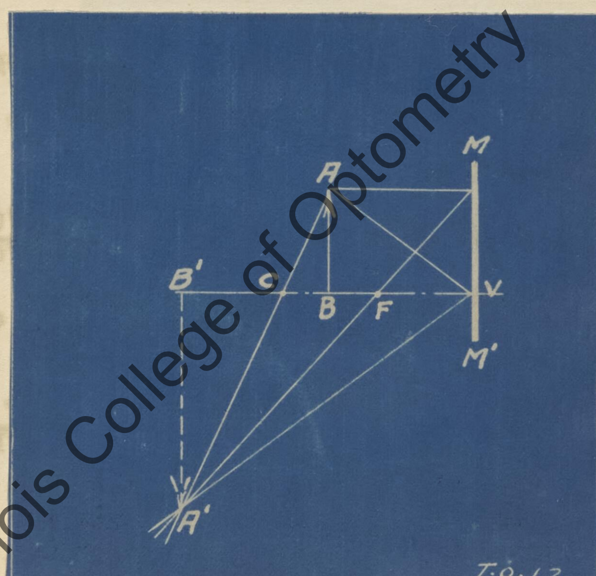
$$\therefore \frac{AV}{II'} = \frac{VF}{I'F}$$

Since AV = OO'

$$\frac{O'C}{I'C} = \frac{VF}{I'F}$$

$$\frac{f_1 + 2F}{2F - f_2} = \frac{F}{F - f_2}$$


A'
M
A
C
F
B
V
B'
M'
T.O.12


M
A
B'
C
B
F
V
M'
A'
T.O.12

$$f_1f_2 - Ff_1 - 2Ff_2 + 2F^2 = 2F^2 - Ff_2$$

$$f_1f_2 = Ff_2 + Ff_1$$

$$\frac{f_1f_2}{f_1f_2F} = \frac{Ff_2}{f_1f_2F} + \frac{Ff_1}{f_1f_2F}$$

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$Ff_1 - f_1f_2 + 2F^2 - 2Ff_2 = 2F^2 - Ff_2$$

$$-f_1f_2 = Ff_2 - Ff_1$$

$$\frac{-f_1f_2}{f_1f_2F} = \frac{Ff_2}{f_1f_2F} - \frac{Ff_1}{f_1f_2F}$$

$$-\frac{1}{F} = \frac{1}{f_1} - \frac{1}{f_2}$$

It will be noticed that the two equations above are similar in form; differing only in signs. If we reckon all distances in front of the mirrors as positive and those behind as negative, we may use one standard conjugate formula for all mirrors.

$$\boxed{\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}}$$

5.

D. Magnification.

1. Real Image. 2. Virtual Image.

The ratio of the linear dimensions of the image and object is termed the magnification.

$$M = \frac{A'B'}{A\,B}$$

When the image is erect, the object and image are on the same side of the axis and both have similar signs, so the magnification is positive.

When the image is inverted, the object and image are on different sides of the axis, so that if one is positive, the other must be negative, and the magnification is negative.

?

Non-numerical

Case 1.

$$M = - \frac{A'B'}{A\,B}$$

Case 2.

$$M = + \frac{A'B'}{A\,B}$$

Now:-

Since $\angle AVB = \angle A'VB$, the right triangles ABV and A'B'V are similar.

$$\therefore \frac{A'B'}{A\,B} = \frac{B'V}{B\,V}$$

$$\frac{\text{Size of Image}}{\text{Size of Object}} = \frac{\text{Distance of Image } (f_2)}{\text{Distance of Object } (f_1)}$$

Making allowance for the difference in signs depending upon the kinds of images, we may use one standard formula for magnification.

$$\boxed{M = \frac{SI}{SO} = \frac{DI}{DO}}$$

6.

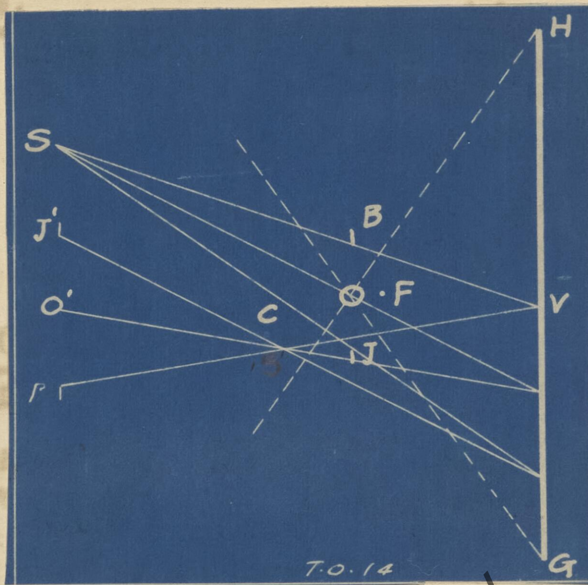
E. Field of View.

1. General.

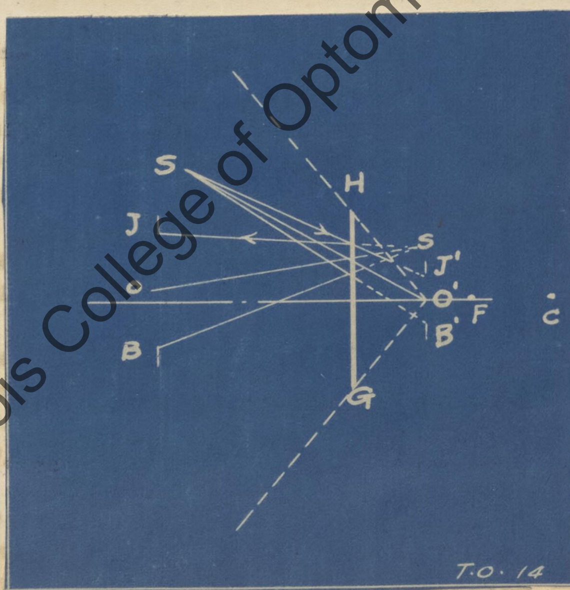
When the image of a luminous object is viewed in a spherical mirror, on the assumption that the image is formed by the reflection of paraxial rays only, the actual portion of the mirror that is utilized consists of a small circular zone immediately surrounding the vertex V. A principal section of this zone is represented by the line GH.

O' represents the center of the observer's pupil and O is its conjugate. The limit of the effective field of view of the mirror is represented by the lines OH and OG (produced if necessary).

To be visible to an eye at O' all object points must lie within the surface of a right circular cone generated by the revolution of the isosceles triangle OHG about the axis of



H
S
B
J'
O'
O
F
C
V
J
P
T.O. 14
G



S
H
J
S
J'
O'
F
C
B
B'
G
T.O. 14

the mirror.

The contour of the effective portion of the spherical mirror acts, as in the case of a plane mirror, as a field-stop for the imagery produced by paraxial rays.

Convex and Concave.

2. Graphical Representation.
   a. Preliminaries.
      (1) O' represents the center of the observer's pupil.
      (2) J' and B' are the outside limits of that pupil.
      (3) O'V is the axis and HVG the effective portion of the mirror.
      (4) S is the point-source of light.
   b. Construction.
      (1) JOB is conjugate to J'O'B'; located by the accepted method of constructing images.
      (2) Join S to J,O, and B.
      (3) Join the points where the above light rays cross the mirror, to their corresponding conjugates J', O' and B'.
      (4) The point S' where the last three reflected rays intersect (produced if necessary), is the conjugate of S.

F. Problems.
1. Show that a plane mirror bisects at right angles the line joining an object-point with its image.
2. A test-chart is 12 feet from a plane mirror and the patient is 8 feet from the mirror. What is the least possible size of the mirror in which the patient may see a full sized reflection of the chart?
3. What must be the length of a vertical, plane mirror in order that a man standing in front of it may see a full length image of himself?
4. Show that the images of a luminous object placed between two plane, inclined mirrors all lie on a circle.

5. How many images of an object-point can be seen in two plane mirrors inclined at the following angles:-

   (a) 90°; (b) 72°; (c) 60°; (d) 45°.

6. If, when standing between two plane, inclined mirrors, a man sees 4 images of himself, at what angle were the mirrors inclined?
7. Two plane mirrors are inclined at an angle of 50°. Show that there will be 7 or 8 images of a luminous point placed between them, according as its angular distance from the nearer mirror is or is not less than 20°.
8. A ray of light is reflected at a plane mirror. Show that if the mirror is turned through an angle θ, the reflected ray will be turned through an angle 2θ.
9. If a plane mirror is turned through an angle of 5°, what is the deflection indicated by the reading on a straight scale 100 cms. from the mirror?
10. Find the angle turned through by a mirror when the deflection on a straight scale 100 cms. away is 10 cms.
11. The top of a vertical plane mirror 2 feet high is 4 feet from the floor. The eye of a person standing in front of the mirror is 6 feet from the floor and 3 feet from the mirror. What are the distances from the wall on which the mirror hangs of the farthest and nearest points on the floor that are visible in the mirror?
12. Describe how the dihedral angle of a glass prism is measured on a goniometer-circle.
13. The radius of a concave mirror is 60 cms. A luminous point is placed in front of the mirror at a distance of:-

    (a) 120 cms; (b) 60 cms; (d) 30 cms; (e) 20 cms.

    Where is the position of the image point?
14. An object is placed 1 foot from a concave mirror of radius 4 feet. If the object is moved 1

inch nearer the mirror, what will be the corresponding displacement of the image?

15. How far from a concave mirror of focal length 18" must an object be placed in order that the image shall be magnified 3 times?
16. Illustrate by diagrams, the formation of virtual images by:-

    (a) Plane; (b) Convex; (c) Concave mirrors.
17. A gas-flame is 8 ft. from a wall, and it is required to throw on the wall a real image of the flame which shall be magnified 3 times. Determine the position and focal length of the concave mirror which would give the required image?
18. What is the magnification in each case, where an object is placed 12 inches in front of a mirror of 18 inches focal length; if the mirror is:-

    (a) Convex
    (b) Concave
19. A concave and a convex mirror, each of radius 20 cms., are placed opposite to each other and 40 cms. apart on the same axis. An object 3 cms. high is placed midway between them. Find the position and size of the image formed by reflection:-
    (a) First, at the convex, and then at the concave mirror.
    (b) First, at the concave, and then at the convex mirror.
20. An object is placed in front of a concave mirror at a distance of one foot. If the image is 3 times the size of the object, what is the focal length of the mirror?

CHAPTER II.

REFRACTION.

I. Plane Surface.

A. Snell's Law.

1. Velocity and Refraction.

Refraction of light in passing obliquely from one medium to another of different density is due to the fact that the speed of light in the one medium differs from its speed in the other medium.

When the velocity of light in air is compared with its velocity in a denser medium the result is a number which is called the index of refraction of the denser medium. This number is really a ratio between the index of the denser medium and the index of air, which is unity.*

The relation between light-speed and index of refraction may be expressed as follows:-

$$\frac{\text{Velocity in 1st medium}}{\text{Velocity in 2nd medium}} = \frac{\text{index of 2nd medium}}{\text{index of 1st medium}}$$

X
7

$$\boxed{\frac{V_1}{V_2} = \frac{u_2}{u_1}}$$

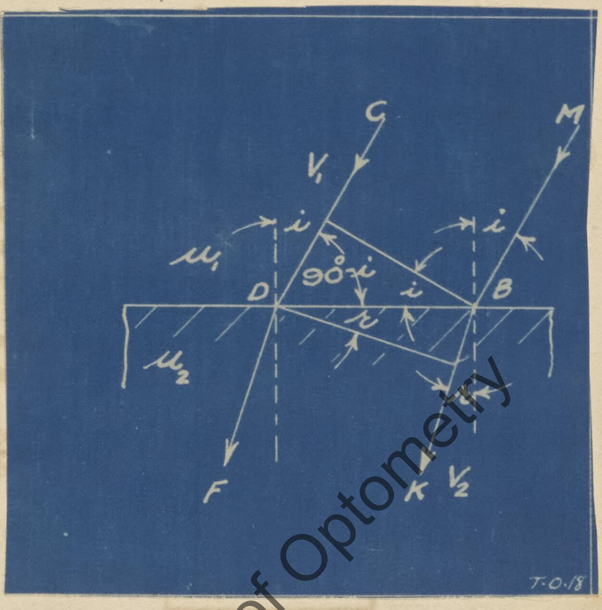
"The ratio of the velocities of light in a pair of media is equal to the ratio of the reciprocals of the indices of refraction of the respective media."

(Note:- When air is one of the media, u_1 = 1 and the formula becomes

$$\frac{V_1}{V_2} = u.)$$

---

* The index of refraction of a vacuum is unity; that of air is 1.0003. For all practical purposes we may take the index of air as being the same as that of a vacuum.


G
M
V_1
i
i
μ_1
90°-i
i
D
B
r
μ_2
F
K
V_2
T.O.18

2. Geometrical Construction.
   a. Explanation.
      (1) MB and CD represent a beam of light travelling with a velocity V_1 in a medium of index u_1.
      (2) BL, perpendicular to MB and CD. represents a wave-front.
      (3) BK and DF represent the refracted beam travelling with a velocity V_2 in a medium of index u_2.
      (4) DR, perpendicular to DF and BK, represents a wave-front.
   b. Proof.

At the instant that the wave-front at B enters the denser medium, the point L, on the same wave-front, has still to travel the distance LD.

During the time that the wave-front travels from L to D, the part at B has travelled from B to R in the denser medium.

Now:-

Distance = velocity x time.

$$\therefore LD = V_1.T$$

$$\text{and } BR = V_2T$$

But:-

$$\frac{LD}{BD} = \text{Cos}(90-i) \text{ and } LD = BD\ \text{Cos}(90-i) =$$

$$= BD\ \text{Sin } i$$

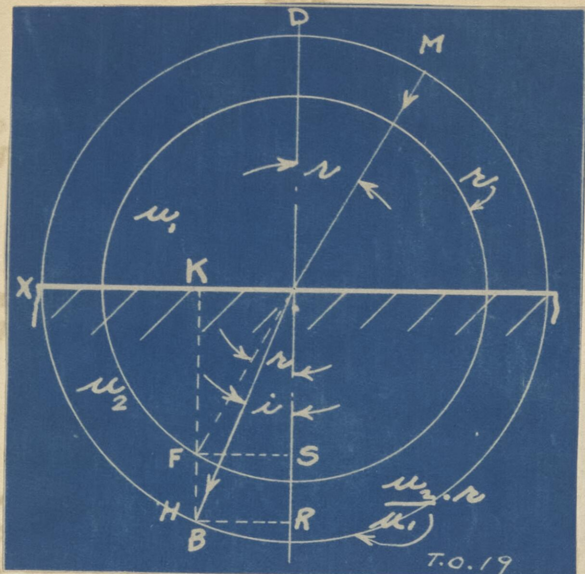
$$\frac{BR}{BD} = \text{Sin } r \text{ and } BR = BD\ \text{Sin } r$$

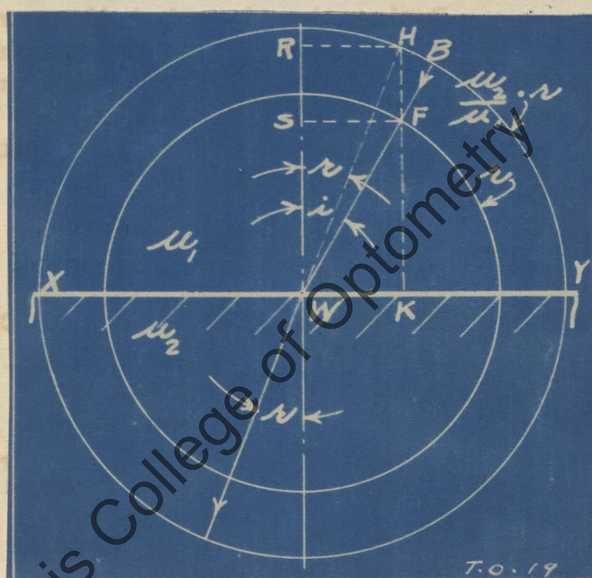
Hence:-

$$\frac{LD}{BR} = \frac{BD\ \text{Sin } i}{BD\ \text{Sin } r} = \frac{V_1.T}{V_2.T}$$

$$\text{and } \frac{\text{Sin } i}{\text{Sin } r} = \frac{V_1}{V_2}$$

$$\text{But } \frac{V_1}{V_2} = \frac{u_2}{u_1}$$


D
M
r
r_1
μ_1
K
X
r
i
μ_2
F
S
H
R
B
$\frac{\mu_2}{\mu_1} \cdot r$
T.O.19


R
H
B
$\frac{\mu_2}{\mu_1} \cdot r$
S
F
r
i
r_1
μ_1
X
Y
W
K
μ_2
r
T.O.19

X
8

$$\therefore \boxed{\frac{\text{Sin } i}{\text{Sin } r} = \frac{u_2}{u_1}}$$

(Notes:-

(1) When air is the less dense medium, $u_1 = 1$ and the formula becomes:-

$$\frac{\text{Sin } i}{\text{Sin } r} = u.$$

(2) When the angles i and r are very small (rays at almost normal incidence), the formula becomes:-

$$\frac{i}{r} = u.)$$

B. Refracted Ray.

1. Construction.

Given an incident ray at a plane surface separating two media of respective indices of refraction u_1 and u_2, it is possible to determine geometrically the direction of the refracted ray.

#1 Rare to Denser

a. Data.

XY represents the refracting surface. BW is the incident ray. RD is the normal at the point W.

b. Method.

(1) With center W and any convenient radius (r), describe a circle, intersecting the ray at F.

(2) With the same center and a radius of $\frac{u_2}{u_1} \cdot r$, describe a second circle.

#2 Dense to Rarer.

Data.

XY represents the refracting surface. BW is the incident ray. RD is the normal at the point W.

Method.

(1) With center W, and any convenient radius r, describe a circle.

(2) With the same center and a radius $\frac{u_2}{u_1} \cdot r$, describe a second circle, intersecting the ray at H.

(3) Through F draw FK normal to the refracting surface and produce FK until it meets the larger circle at H.

(4) HW produced (WM) represents the refracted ray.

2. Proof

From H and F draw HR and FS parallel to the refracting surface.

The angle FWS is the angle of incidence (i).

Now:-

$$\frac{\text{Sin i}}{\text{Sin HWR}} = \frac{FS/FW}{RH/HW}$$

$$= \frac{FS}{FW} \times \frac{HW}{RH}$$

(Since FS = RH)

$$= \frac{HW}{FW}$$

$$= \frac{\frac{u_2 . r}{u_1}}{r}$$

$$= \frac{u_2}{u_1}$$

As this is in accordance with Snell's Law, then HWR = r (angle of refraction); and WM is the refracted ray.

(3) Through H draw HK normal to the refracting surface and let the point where it intersects the smaller circle be F. FW produced (WM) represents the refracted ray.

2. Proof.

From H and F draw HR and FS parallel to the refracting surface.

The angle BWR is the angle of incidence (i).

Now:-

$$\frac{\text{Sin i}}{\text{Sin FWS}} = \frac{HR/HW}{FS/FW}$$

$$= \frac{HR}{HW} \times \frac{FW}{FS}$$

(Since FS = HR)

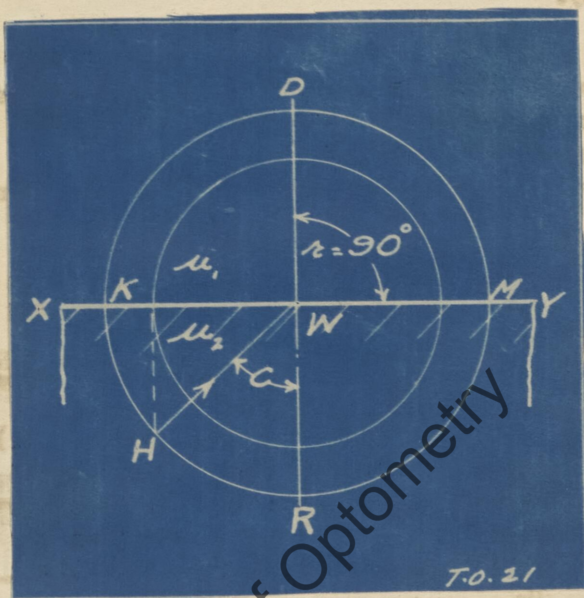
$$= \frac{FW}{HW}$$

$$= \frac{\frac{r}{1}}{\frac{u_2 . r}{u_1}}$$

$$= \frac{r}{1} \times \frac{u_1}{u_2 r}$$

$$= \frac{u_1}{u_2}$$

As this is in accordance with Snell's Law, then MWD = r (angle of refraction); and WM is the refracted ray.


D
$r = 90°$
μ_1
K
M
X
Y
W
μ_2
c
H
R
T.O. 21

C. Critical Angle.

1. Analysis.

In diagram #2 of section B preceding, we see that the position of the normal HFK depends upon the size of the angle of incidence HWR, and that the position of H on the larger circle determines the size of the angle of refraction DWM.

As HWR increases, the point F approaches the point K and the angle DWM increases.

Finally we arrive at a limit where the point F coincides with the point K and HK is tangent to the smaller circle.

When this limit is reached, the point M will be situated on the surface XY, and the direction of the refracted ray will be along KW produced (WM); just grazing the refracting surface. The angle of refraction will then be equal to a right angle.

The angle of incidence HWR, when this limit is reached, is called the critical angle.

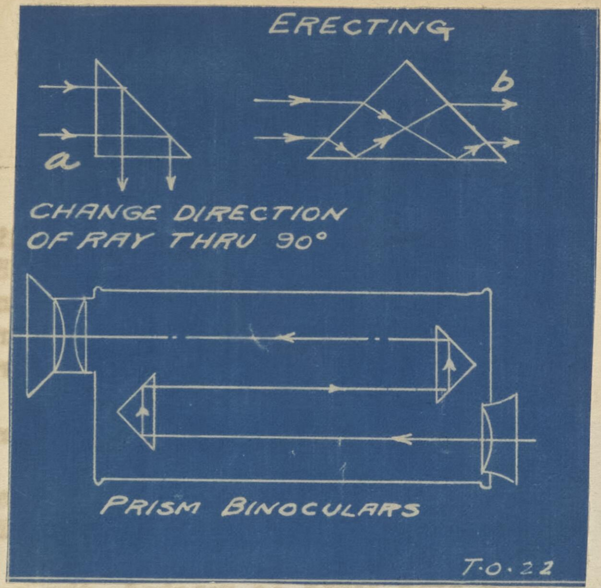
If the angle of incidence in the denser medium is larger than the critical angle, then HK will never meet the inner circle and there will be no angle of refraction because there will be no emergent ray, the light having been totally internally reflected.

2. Calculation.

$$\frac{\text{Sin } i}{\text{Sin } r} = \frac{u_1}{u_2}$$

$$\frac{\text{Sin } C}{\text{Sin } 90^\circ} = \frac{u_1}{u_2}$$

$$\frac{\text{Sin } C}{1} = \frac{u_1}{u_2}$$


ERECTING
a
b
CHANGE DIRECTION
OF RAY THRU 90°
PRISM BINOCULARS
T-O-22

$\therefore$ Sine of the Critical Angle $= \dfrac{\text{lesser index}}{\text{greater index}}$

X
9.

$$\boxed{\text{Sin C} = \frac{u_1}{u_2}}$$

(Note:- When air is one of the media $u_1 = 1$,
and the formula becomes:-

$$\text{Sin C} = \frac{1}{u}.)$$

3. Practical Application.

The principle of total internal reflection is utilized by the employment of totally reflecting prisms. These are rectangular glass prisms whose two other angles are 45° each.

If rays of light are normally incident on either of the two mutually rectangular faces, they will pass into the prism unrefracted and will meet the hypotenuse face at an angle of 45°. This is greater than the critical angle for glass, so the rays will be totally reflected at the hypotenuse face and will pass out of the prism at the other mutually rectangular face.

This remarkable property is put to use to:-

a. Change the direction of a ray through 90°.

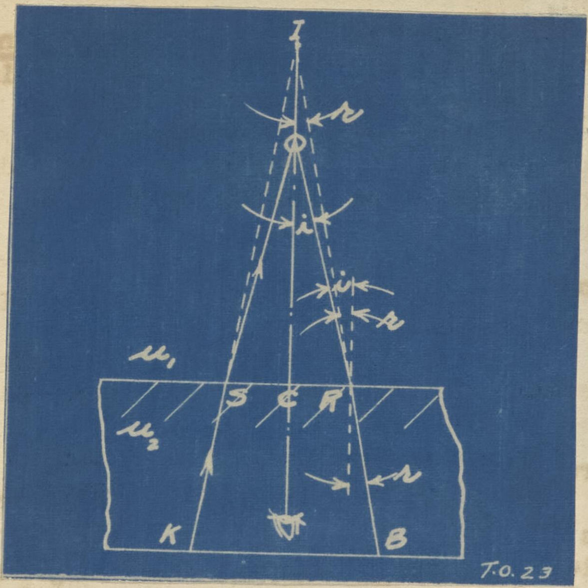
b. Erect an inverted image.

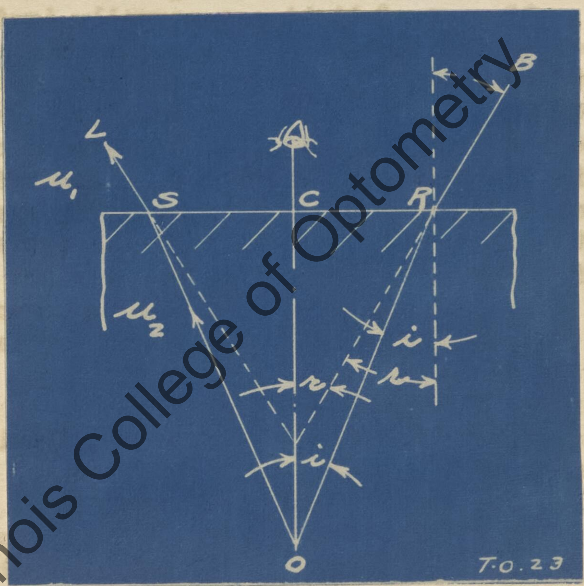
The instruments in which these prisms are used are:-

a. Magic lantern.

b. Prism-binocular field-glasses.

The following diagrams will illustrate their use.


I
r
O
i
i
r
μ_1
S C R
μ_2
r
K
B
T.O.23


B
V
μ_1
S C R
μ_2
i
r r
i
O
T.O.23

D. Refractive Imagery.

1. General.

Looking at almost normal incidence from one medium into another of different index, a person will see an object as if it were displaced. In reality he is looking at an image of the object.

The direction of the displacement will depend upon whether the refraction is towards or away from the normal.

"When a person looks from a rare to a denser medium, objects appear to be nearer than they really are."

"When a person looks from a dense to a rarer medium, objects appear farther away than they really are."

Looking into Denser Medium

2. Specific.
   a. Light from the object follows the paths OR and OS.
   b. In passing from a dense to a rarer medium it is refracted along RB and SK.
   c. To an observing eye the light appears to have come from the point I.
   d. I is the image of O by refraction of rays almost normally incident.

3. Calculation

$$\frac{\text{Sin } i}{\text{Sin } r} = \frac{u_1}{u_2}$$

Sines of small angles are equal to their tangents.

Looking into Rarer Medium.

2. Specific.
   a. Light from the object follows the paths OR and OS.
   b. In passing from a rare to a denser medium it is refracted along RB and SK.
   c. To an observing eye the light appears to have come from the point I.
   d. I is the image of O by refraction of rays almost normally incident.

3. Calculation.

$$\frac{\text{Sin } i}{\text{Sin } r} = \frac{u_2}{u_1}$$

Sines of small angles are equal to their tangents.

$$\therefore \frac{\tan i}{\tan r} = \frac{u_1}{u_2}$$

$$\frac{RC/CO}{RC/CI} = \frac{u_1}{u_2}$$

$$\frac{RC}{CO} \times \frac{CI}{RC} = \frac{u_1}{u_2}$$

$$\frac{CI}{CO} = \frac{u_1}{u_2}$$

$$CI = \frac{u_1}{u_2}.CO$$

Distance of Image $= \frac{u_1}{u_2}$. Distance of Object.

Apparent distance $= \frac{u_1}{u_2}$. Real distance.

$$\boxed{f_2 = \frac{u_1}{u_2}.f_1}$$

(Notes:-

a. Since $u_2 > u_1$; $f_2 < f_1$

The image appears nearer to the eye than the object.

b. If air is one of the media,

$u_1 = 1$ and

$$f_2 = \frac{f_1}{u}$$

$$\therefore \frac{\tan i}{\tan r} = \frac{u_2}{u_1}$$

$$\frac{RC/CO}{RC/CI} = \frac{u_2}{u_1}$$

$$\frac{RC}{CO} \times \frac{CI}{RC} = \frac{u_2}{u_1}$$

$$\frac{CI}{CO} = \frac{u_2}{u_1}$$

$$CI = \frac{u_2}{u_1}.CO$$

Distance of Image $= \frac{u_2}{u_1}$. Distance of Object.

Apparent distance $= \frac{u_2}{u_1}$. Real Distance.

$$\boxed{f_2 = \frac{u_2}{u_1}.f_1}$$

(Notes:-

a. Since $u_2 > u_1$; $f_2 > f_1$

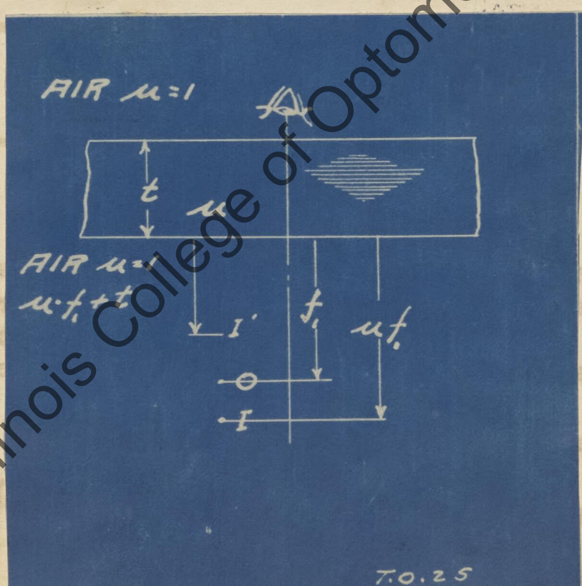
The image appears farther removed from the eye than the object.

b. If air is one of the media,

$u_1 = 1$ and

$f_2 = uf_1$)

X
10


AIR μ=1
t
μ
AIR μ=1
μ·f₁+t
I'
f₁
μf₁
O
I
T.O.25

c. Refractive imagery gives us a simple means of determing accurately the index of a liquid.)

E. Displacement by Refraction.

1. Perpendicular.

a. Plate in Air (Parallel Sides).

There will be two refractions; one at the plane BC and the other at the plane DK.

The refraction at CB will make the image appear farther removed from BC = uf_1.

Distance of I from DK is now (uf_1 + thickness of plate) = (uf_1 + t).

The second refraction, the one at the plane DK, will make I' appear nearer than I = $\dfrac{uf_1 + t}{u}$.

But the actual distance of O from DK = $(f_1 + t)$

Hence, the amount of the total displacement (d) = $(f_1 + t) - \left(\dfrac{uf_1 + t}{u}\right)$

$$= \frac{uf_1 + ut - uf_1 - t}{u}$$

$$= \frac{ut - t}{u}$$

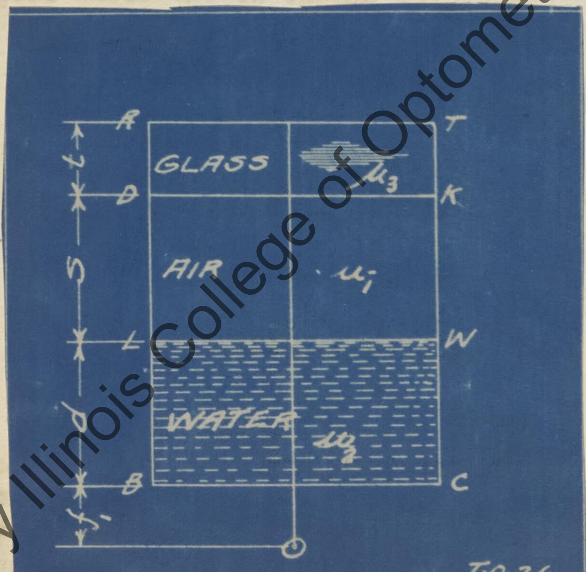
$$= \frac{t\,(u-1)}{u}$$

$$\boxed{d = \frac{t\,(u-1)}{u}}$$

(Note:- This displacement is proportional to the thickness of the plate, and by making "t" as small as possible, we can make the displacement as small as we desire.)

X

11


A
T
t
GLASS
u_3
D
K
S
AIR
u_1
L
W
d
WATER
u_2
B
C
f_1
O
T.O.26

X b. Multiple Refractions.

There are four refracting surfaces; BC, VW, DK and RT. The object "O" which is at a distance f_1 from the bottom of the vessel, is separated from the free surface RT by a distance f_1; a depth of water (d); an air space (s); and a thickness of glass (t).

The position of the final image may be calculated as follows:-

(1) Refraction at BC will make the image of O appear farther removed to a distance = u_2f_1 and its distance from VW is now $(u_2f_1 + d)$.

(2) Refraction at VW will bring the image nearer, to a distance = $\frac{u_2f_1 + d}{u_2}$ and its distance from DK = $\left(\frac{u_2f_1 + d}{u_2} + s\right) =$

$$\left(\frac{u_2f_1 + d + u_2s}{u_2}\right).$$

(3) Refraction at DK will make the image appear farther removed to a distance =

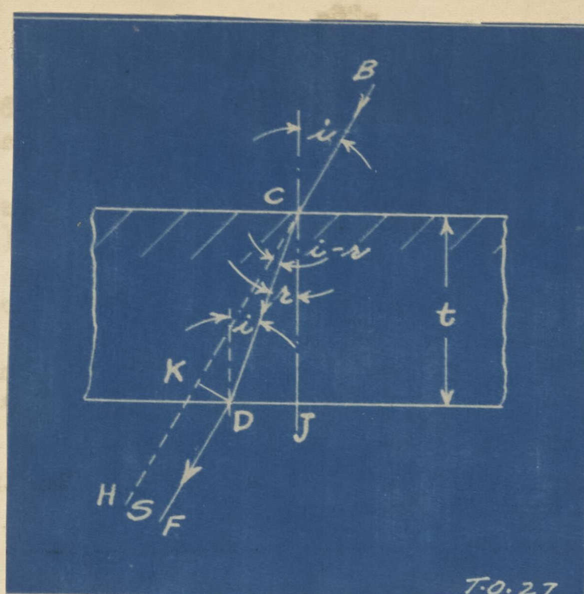
$$\left(\frac{u_2f_1 + d + u_2s}{u_2}\right) \times u_3$$

$$= \left(\frac{u_2u_3f_1 + u_3d + u_2u_3s}{u_2}\right)$$ and its distance from the free-surface RT =

$$\left(\frac{u_2u_3f_1 + u_3d + u_2u_3s}{u_2}\right) + t$$

$$= \left(\frac{u_2u_3f_1 + u_3d + u_2u_3s + u_2t}{u_2}\right).$$

(4) Refraction at the free surface RT will bring the final image nearer, to a distance =


B
i
C
i-r
r
t
i
K
D
J
H
S
F
T-O.27

$$= \left(\frac{u_2u_3f_1 + u_3d + u_2u_3s + u_2t}{u_2}\right) \div u_3$$

$$= \frac{u_2u_3f_1 + u_3d + u_2u_3s + u_2t}{u_2u_3}$$

$$= \frac{f_1}{1} + \frac{d}{u_2} + \frac{s}{1} + \frac{t}{u_3}.$$

Rule:- "Divide the thickness of each refracting medium by its own index and sum the series to obtain the distance of the final image from the free surface."

X 12

$$\boxed{f_2 = \frac{t_1}{u_1} + \frac{t_2}{u_2} + \frac{t_3}{u_3} + - - - - + \frac{t_n}{u_n}}$$

2. Lateral.

a. Explanation.

BC is the incident ray, which after refraction at each of the parallel surfaces of the plate, emerges along DF parallel to its original direction along the line BCKH (vide I; F).

There has been an obvious displacement which is measured by the distance KD, which is perpendicular to both the final and original light paths.

CJ is normal to the surfaces of the plate and is also equal to the thickness (t).

b. Calculation.

The triangles CDK and CDJ have a side CD in common.

$$\frac{DK}{CD} = \text{Sin } (i-r) \quad \text{and CD} = \frac{DK}{\text{Sin}(i-r)}$$

$$\frac{CJ}{CD} = \text{Cos r. and CD} = \frac{CJ}{\text{Cos.r}}$$

A.

$-15 \bigcirc +8$ ax 20, Find power in 170°
$-7 \bigcirc -8$ ax 110°

$$\begin{array}{r} 170 \\ 110 \\ \hline 60 \end{array} \qquad \frac{60}{90} = \frac{2}{3}$$

$$\frac{2}{3}(-8) = \frac{-16}{3} = -5\tfrac{1}{3}$$

$$\begin{array}{r} -7.00 \\ -5.33 \\ \hline -12.33 \end{array} \text{ ax } 170$$

---

1. A patient has 10^Δ exotropia
Rx { O.D. $+5 \bigcirc +6$ ax 60
{ OS $+4 \bigcirc -3$ ax 120

It is desired to correct the exotropia in full by decentering the lenses placing 2/3 of the prismatic correction over the rt. eye and the remaining in the left.

2. Patient has 12^Δ of esotropia. Correct full amount by decentering. Place 3/4 of the prismatic over left eye and remaining over right.
Rx { O.D. $+1 \bigcirc +3$ ax 30
{ OS $+8 \bigcirc -3$ ax 60

3. Patient has 6^Δ of esophoria. Only 2/3 is to be corrected, by decentering equal amounts of prism over each eye. Rx OD { $-6 \bigcirc -3$ ax 120
OS { $-6 \bigcirc -6$ ax 30

$$\frac{2}{3}\cdot \cancel{6} \cdot 4 + 5 = 9\,D. \qquad \Delta = Dd \qquad \frac{2}{3}\cdot -3 = \begin{array}{r} -2 \\ +4 \\ \hline +2 \end{array}$$

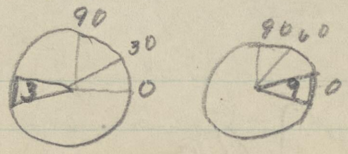
$$\frac{2}{3}\,10 = \frac{20}{3} \qquad 9c = \frac{20}{3}$$

$$c = \frac{20}{27} = 7.4\text{ mm}$$ decentered in.

$2x = 3.33$
$x = 1.6$
$x = 16$ mm in

$$\begin{array}{r} 7.4 \\ 27 \overline{)\,200} \\ 189 \\ \hline 110 \\ 108 \end{array}$$

$+1 \supset +3 \text{ ax } 30$ OD OS $+8 \supset -3 \text{ ax } 60$



$\frac{1}{3} \cdot 3 + 1 = 2\text{D}$

$2x = 3$

$x = 1\frac{1}{2}$ cm

15 mm

4244 Drexel

5130 Kenwood Apt 302 Mr Kelli

Ken 2397 Hoffe

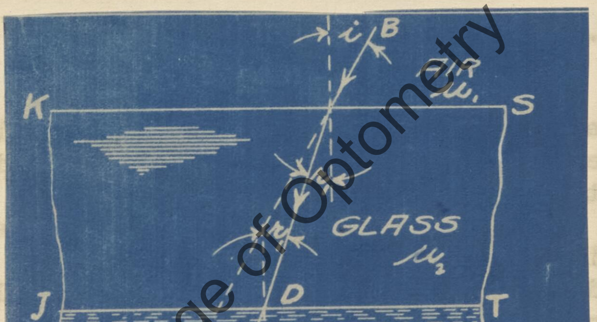
$24 \times 9 = 216.$

$216. - 168. = 48.$

$2680.00 + 3888.00 = 7168.00$

$350. \div 4 = 87.50$

$200. \times 3\frac{1}{2} = 700.$



i
B
K
S
GLASS μ_2
J
D
T
WATER μ_3
F
W
AIR μ_1
A
H
T.O.28

$$\therefore \frac{DK}{\operatorname{Sin}(i-r)} = \frac{CJ}{\operatorname{Cos} r}$$

$$DK = \frac{\operatorname{Sin}(i-r)}{\operatorname{Cos} r.} \cdot CJ$$

$$\boxed{d = \frac{\operatorname{Sin}(i-r)}{\operatorname{Cos} r} \cdot t.}$$

13

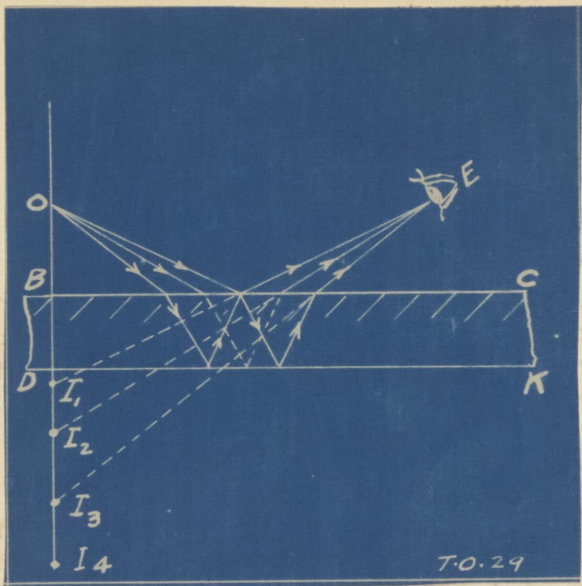
"The lateral displacement is proportional to the thickness of the plate, and by varying this thickness we can vary the amount of displacement as we desire."

F. Compound Plate.

"When light passes through a series of refracting media, with adjacent, plane, parallel surfaces, no matter what refractions have taken place, the emergent ray will be parallel to the incident ray if the medium containing the emergent ray is the same, or has the same index as that containing the incident ray."

1. Explanation.
   BCDFH is the path of a ray which successively traverses two parallel plates with a common interface JT. The first medium is the same as the last, having an index of u_1.
   AH can be proved to be parallel to BC.
   The angles of incidence, and the angles of refraction at each intersurface are indicated in the diagram, together with the indices of refraction of each medium.
   If the angle i = the angle A, then FH is parallel to BC.
2. Proof.
   By Snell's Law:-

$$\frac{\operatorname{Sin}\ i}{\operatorname{Sin} r_1} = \frac{u_2}{u_1} \text{ at KS}$$


E
O
B
C
D
K
I_1
I_2
I_3
I_4
T.O.29

$$\frac{\text{Sin } r_1}{\text{Sin } r_2} = \frac{u_3}{u_2} \quad \text{at JT}$$

$$\frac{\text{Sin } r_2}{\text{Sin A}} = \frac{u_1}{u_3} \quad \text{at VW}$$

$$\therefore \quad \frac{\text{Sin i}}{\text{Sin } r_1} \times \frac{\text{Sin } r_1}{\text{Sin } r_2} \times \frac{\text{Sin } r_2}{\text{Sin A}} = \frac{u_2}{u_1} \times \frac{u_3}{u_2} \times \frac{u_1}{u_3}$$

$$\frac{\text{Sin i}}{\text{Sin A}} = 1$$

$$\text{Sin i} = \text{Sin A}$$

$$\therefore \quad i = A$$

BC and HF are parallel.

X G. Multiple Images.

When there is but one reflecting surface, as in a metal mirror, there will be but one image. In a glass mirror, having two reflecting surfaces, namely, the front surface of the glass (BC), and the silvered back surface (DK), there are multiple images of an object.

If a light source "O" be held near to the glass mirror, a series of images will be seen by the eye at E.

The first image I_1 is that which is formed by the front unsilvered surface (BC).

The second image I_2 which is the brightest, is directly reflected from the back, silvered surface.

The other images, equally distant from each other, are formed by repeated internal reflection, some light escaping after each reflection.

The images become progressively fainter, the number of images depending upon the intensity of the light.


μ1/μ2 · N
μ=1
T
J
P
V
K
Q
A
M
N
C
Z
F
μ2
R
X
Y
T.O. 30

II. Inclined Surfaces (Prisms)

A. Ray-Path (Graphical).

1. Explanation.

VXY represents the principal section of a prism.
BC is the incident ray making an angle i_1 with the normal NR.
It is required to construct graphically the path of the ray through the prism.

2. Construction.

a. Produce the side XV any convenient length (say to T).
b. With center V and any convenient radius r, describe a circle around the vertex.
c. With the same center and a radius $= \frac{u_1}{u_2} . r$ describe a second circle around the vertex.
d. From V, draw VP parallel to the incident ray BC and meeting the inner circle at P.
e. Through P, draw JP normal to the side XV produced and produce JP until it meets the larger circle at K.
(VK is parallel to the path of the ray inside the prism.)
f. Construct CD parallel to VK.
g. From K, draw KM normal to the side VY and intersecting the smaller circle at Q.
h. Join VQ.
(VQ is parallel to the emergent ray.)
i. Construct DF parallel to VQ.

3. Proof.

VP is parallel to BC.
JK is parallel to NR.

$\therefore$ JPV $= i_1$

Now:-

$$\frac{\text{Sin } i_1}{\text{Sin JKV}} = \frac{VJ/VP}{VJ/VK}$$

$$= \frac{VJ}{VP} \times \frac{VK}{VJ}$$

$$= \frac{VK}{VP}$$

$$= \frac{r}{\frac{u_1}{u_2} \cdot r}$$

$$= \frac{r}{1} \times \frac{u_2}{u_1 r}$$

$$= \frac{u_2}{u_1} .$$

This is in accordance with Snell's Law and since JVP = i_1 then JKV = r_1 (angle of refraction).

Hence, VK is parallel to the path of the ray through the prism.

Now:-

VK is parallel to CD
KM is parallel to ZR

$\therefore$ VKM = r_2

But:-

$$\frac{\text{Sin } r_2}{\text{Sin VQM}} = \frac{VM/VK}{VM/VQ}$$

$$= \frac{VM}{VK} \times \frac{VQ}{VM}$$

$$= \frac{VQ}{VK}$$

$$= \frac{\frac{u_1}{u_2} \cdot r}{r}$$

$$= \frac{u_1}{u_2} .$$

This is in accordance with Snell's Law and since VKM = r_2, then VQM = i_2 (angle of emergent refraction.

Hence VQ is parallel to the path of the ray which will emerge from the prism.

B. Analysis.

1. Grazing Incidence.

The position of the point P depends upon the slope of VP, which in turn depends upon the angle of incidence i_1.

As i_1 increases, P will move along the circle, drawing nearer to J and finally will coincide with J.

JK will then be tangent to the inner circle and the incident ray will be parallel to VJ, that is, BC will then be grazing the surface XV.

When this limit is reached, r_1 and VKJ will each be equal to the critical angle (C) for the two media.

2. Grazing Emergence.

As i_1 decreases, Q will move nearer M and finally will coincide with M.

The emergent ray will then be parallel with VM, that is, DF will be grazing the surface VY.

When this limit is reached, r_2 and VKM will each be equal to the critical angle (C) for the two media.

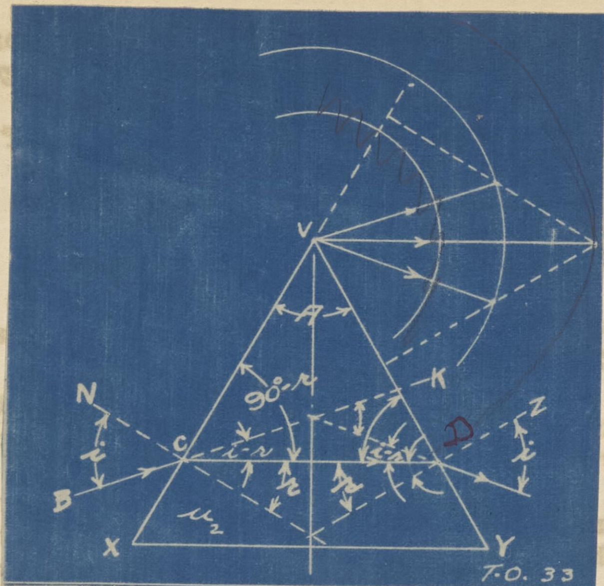
3. Grazing Incidence and Emergence.

There is a condition, depending upon the size of the apical angle (A) of the prism, when both JK and MK will be tangent to the smaller circle. Then, the incident and emergent rays will be parallel respectively to VJ and VM, that is, there will be both grazing incidence and grazing emergence at the same time.

When this limit is reached, the apical angle A will be equal to twice the critical angle for the two media (A = 2C).

4. Minimum Deviation.

There is an intermediate position of the


V
A
90°-r
K
N
C
i-r
Z
B
X
Y
T-O. 33

point K. where KP = KQ. This is when the line VK bisects the angle YVT.

VK is then parallel to the base XY, and VPJ is then equal to VQM.

Under these conditions, the deviation produced by the prism is at its lowest (minimum deviation).

Then $i_1 = i_2$; $r_1 = r_2$; and CD is parallel to XY.

(Note:-

It is an instructive exercise to construct graphically diagrams illustrating the conditions mentioned in the foregoing section.)

C. Minimum Deviation.

1. Explanation.

The principle of the reversibility of light-path demonstrates that when a ray of light passes through a prism set for minimum deviation, that ray will traverse the prism symmetrically.

$$\therefore \quad BCN = FDZ$$

$$RCD = RDC.$$

2. Deductions.

a. Part I.

$$d = (i-r) + (i-r)$$

$$= 2i - 2r$$

$$2i = d + 2r$$

$$i = \frac{d + 2r}{2}$$

b. Part II.

$$A + (VCD) + (VDC) = 180^\circ$$

$$A + (90-r) + (90-r) = 180^\circ$$

$$A + 2(90-r) = 180^\circ$$

$$A + 180-2r = 180^\circ$$

$$A - 2r = 0$$

$$A = 2r$$

$$r = \frac{A}{2}$$

$$\therefore i = \frac{d + 2r}{2} = \frac{d + A}{2} = \frac{A + d}{2}$$

c. Part III.

$$\frac{\text{Sin } i}{\text{Sin } r} = \frac{u_2}{u_1}$$

$$\frac{u_2}{u_1} = \frac{\text{Sin } i}{\text{Sin } r}$$

$$\frac{u_2}{u_1} = \frac{\text{Sin } (\frac{A + d}{2})}{\text{Sin } (\frac{A}{2})}.$$

and when air is the surrounding mediu, as in the case of ophthalmic prisms, $u_1 = 1$ and the formula becomes:-

$$\boxed{u = \frac{\text{Sin } (\frac{A + d}{2})}{\text{Sin } (\frac{A}{2})}}$$

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D. Acute-Angled Prisms.

When dealing with small-angled prisms ($A = 10^\circ$ or less), we may omit the sines. The formula in the preceding section then becomes:-

$$u = \frac{(\frac{A + d}{2})}{(\frac{A}{2})}$$

$$\frac{u}{1} = \frac{A + d}{2} \times \frac{2}{A}$$

$$\frac{u}{1} = \frac{A + d}{A}$$

$$uA = A + d$$

$$uA - A = d$$

$$A(u-1) = d$$

$$\boxed{d = (u-1)\,A}$$

X 15

E. Degrees, Centrads and Prism-Diopters.

For small angles, Centrads and Prism-Diopters are approximately equal.

$$1 \text{ Centrad} = \frac{1 \text{ Radian}}{100}$$

$$2\pi \text{ Radians} = 360$$

$$1 \text{ Radian} = \frac{360}{2\pi} = \frac{180}{\pi} = 57.3^\circ$$

$$\frac{1 \text{ Radian}}{100} = \frac{57.3^\circ}{100} = .573^\circ$$

If $.573^\circ = 1$ Centrad or 1 P.D.

then $1^\circ = \frac{1}{.573}$ P.D. $= 1.745$ P.D.

But:-

$$d = (u-1)\,A^\circ$$

$$\therefore d \text{ (Centrads)} = 1.745\,(u-1)\,A.$$

$$\boxed{\text{P.D.} = 1.745(u-1)\,A}$$

X 16

F. Resultant Prisms.

1. Arithmetical.

a. Amount

(1) In optometric practice we are dealing with a combination of prisms in which the component prisms are always at right angles.

(2) Prism powers, like mechanical forces, can be represented by straight lines drawn

to scale, so that a combination of prisms in an ophthalmic prescription is based upon the mathematics of a right triangle.

(3) Resultant Prism = $\sqrt{(\text{1st P D.})^2 + (\text{2nd P.D})^2}$ X

$$\boxed{P_R = \sqrt{P_1^{\,2} + P_2^{\,2}}}$$ 17

b. Base - Apex Line.

(1) The base-apex line of the resultant prism will always be between the base-apex lines of the two component prisms, that is, between the horizontal and the vertical.

(2) The angle of inclination which the base-apex line of the resultant prism makes, with the base-apex line of each separate component, is that part of 90° expressed by the ratio of each prism separately to the combined sum of both.

Thus:-

X

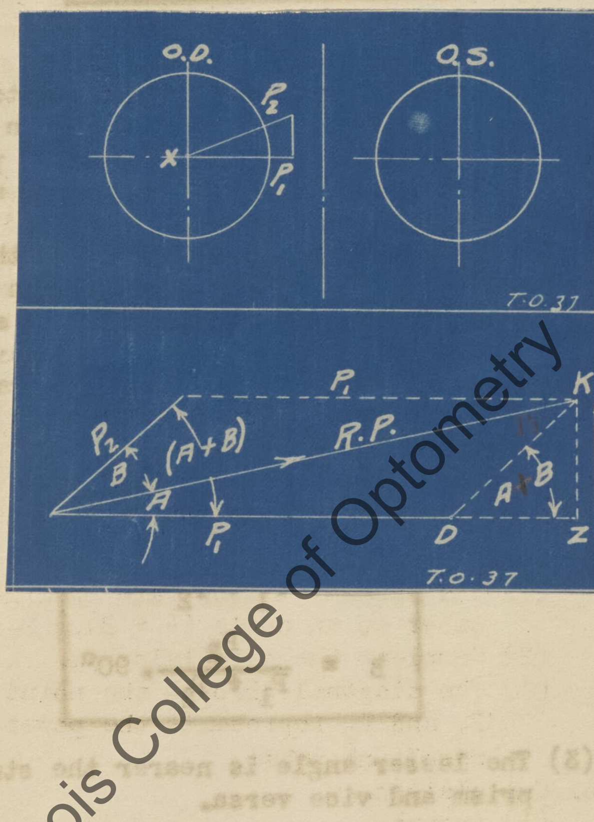
$$\boxed{\begin{aligned} A &= \frac{P_1}{P_1 + P_2} \cdot 90^\circ \\ B &= \frac{P_2}{P_1 + P_2} \cdot 90^\circ \end{aligned}}$$ 18

(3) The lesser angle is nearer the stronger prism and vice versa.

2. Graphical.

a. Amount.

(1) Divide the right and left fields (patient facing you) by a vertical line (median line).


O.D.
O.S.
P_2
P_1
T.O. 37
P_1
K
P_2
R.P.
(A+B)
B
A
A+B
P_1
D
Z
T.O. 37

(2) Mark a dot to represent the center of the pupil of the eye under consideration.

(3) Draw a horizontal line to scale (1cm = 1 P.D.), in the direction of the base of the horizontal component.

(4) At the "base" extremity of the above line, erect a perpendicular to scale (1 cm = 1 P.D.), in the direction of the base of the other component.

(5) The sloping line connecting the "free" end of each of the two component lines will give the power of the resultant prism if measured in the same units as the scale.

b. Base-Apex Line.

The angle which the hypotenuse makes with the horizontal component, measured with a protractor, will give the direction of the base-apex line of the resultant prism when compared with the horizontal.

3. Trigonometrically.

By means of trigonometry, we may calculate the power of the resultant prism and determine the slope of its base-apex line even if the component prisms are not at right angles.

a. Data.

(1) The base-apex lines are inclined to each other at an angle (A + B).

(2) Complete the parallelogram.

(3) The diagonal through the point common to P_1 and P_2 represents the power of the resultant prism.

b. Calculation.

(1) Amount.

$$CK = \sqrt{CZ^2 + ZK^2}$$

$$= \sqrt{(CD + DZ)^2 + ZK^2}$$

But:-

$$\frac{DZ}{P_2} = \text{Cos }(\overline{A + B}) \text{ and } DZ = P_2 \text{ Cos }(\overline{A + B})$$

$$\frac{ZK}{P_2} = \text{Sin }(\overline{A + B}) \text{ and } ZK = P_2 \text{ Sin }(\overline{A + B})$$

$$\therefore \quad CK = \sqrt{(P_1 + P_2 \text{ Cos }(\overline{A + B}))^2 + (P_2 \text{ Sin }(\overline{A + B}))^2}$$

$$= \sqrt{P_1^{\,2} + 2P_1P_2\text{Cos}(\overline{A + B}) + P_2^{\,2}\text{Cos}^2(\overline{A + B}) + P_2^{\,2}\text{ Sin}^2\ (\overline{A + B})}$$

$$= \sqrt{P_1^{\,2} + 2P_1P_2\text{Cos}(\overline{A + B}) + P_2^{\,2}(\text{Cos}^2(\overline{A + B}) + \text{Sin}^2(\overline{A + B}))}$$

$$= \sqrt{P_1^{\,2} + P_2^{\,2} + 2\ P_1P_2\text{Cos }(\overline{A + B})}\ *$$

$$\boxed{P_R = \sqrt{P_1^{\,2} + P_2^{\,2} + 2P_1P_2\text{Cos }(\overline{A + B})}}$$

(Note:- When $(\overline{A + B}) = 90°$, Cos $(\overline{A + B}) = 0$; and the formula becomes:-

$$P_R = \sqrt{P_1^{\,2} + P_2^{\,2}} \text{ (vide supra) }).$$

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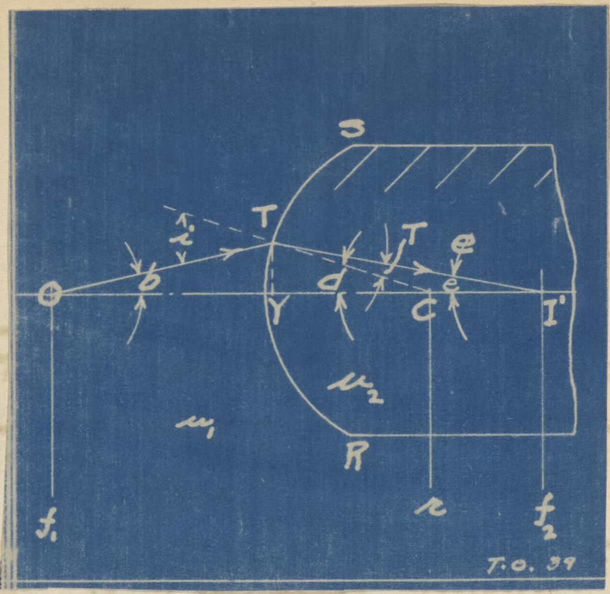
* Cos $(\overline{A + B})$ does not mean the cosine of the sum of two angles A and B; but it does mean the cosine of a single angle whose value is equal to the sum of A and B.


S
T
T
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Y
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I'
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T.O. 39


S
N
T
b
e
Y
V
μ2
μ1
R
f1
f2
r
T.O. 39

(2) Base-Apex Line.

$$\tan A = \frac{ZK}{ZC}$$

$$= \frac{ZK}{(ZD + DC)}$$

$$= \frac{P_2 \text{ Sin } (\overline{A + B})}{P_2 \text{ Cos } (A+B)+P_1}$$

$$\boxed{A = \tan^{-1} \frac{P_2 \text{ Sin } (\overline{A + B})}{P_1+P_2\text{Cos } (\overline{A + B})}}$$

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(Note:-

When $(\overline{A + B}) = 90^\circ$; Sin $(\overline{A + B}) = 1$ and Cos $(\overline{A + B}) = 0$; and the formula becomes:-

$$A = \tan^{-1} \frac{P_2}{P_1} = \frac{\text{Vertical Component}}{\text{Horizontal Component.}}$$)

4. Division of Resultant Prism.
   a. A resultant prism is usually divided equally between the two eyes.
   b. The base-apex line of the one prism is exactly 180° away from the base-apex line of the other (diametrically opposite).

III. Curved Surface (Single Refracting Surface).

A. Abscissa Equation.

1. Data
   OT is the incident ray.
   SVR is the spherical refracting surface (Cx).
   TC is the normal to the surface.
   TI is the refracted ray.

1. Data.
   OT is the incident ray.
   SVR is the spherical refracting surface (Cc).
   TC is the normal to the surface.
   TI is the refracted ray produced.

Angles, distances and indices are marked on the diagram.

2. Proof.

$$\frac{\text{Sin } i}{\text{Sin } r} = \frac{u_2}{u_1}$$

$$u_1 \text{ Sin } i = u_2 \text{ Sin } r$$

$$u_1 \text{ Sin } (b+d) = u_2 \text{ Sin } (d-e)$$

For small angles we may omit the sines.

$$u_1 (b+d) = u_2 (d-e)$$

Substituting the tan.

$$u_1 \left(\frac{TY}{YO} + \frac{TY}{YC}\right) = u_2 \left(\frac{TY}{YC} - \frac{TY}{YI}\right)$$

$$u_1 \left(\frac{1}{YO} + \frac{1}{YC}\right) = u_2 \left(\frac{1}{YC} - \frac{1}{YI}\right)$$

For paraxial rays, Y and V are coincident.

$$u_1 \left(\frac{1}{VO} + \frac{1}{VC}\right) = u_2 \left(\frac{1}{VC} - \frac{1}{VI}\right)$$

$$u_1 \left(\frac{1}{f_1} + \frac{1}{r}\right) = u_2 \left(\frac{1}{r} - \frac{1}{f_2}\right)$$

$$\frac{u_1}{f_1} + \frac{u_1}{r} = \frac{u_2}{r} - \frac{u_2}{f_2}$$

Angles, distances and indices are marked on the diagram.

2. Proof.

$$\frac{\text{Sin } i}{\text{Sin } r} = \frac{u_2}{u_1}$$

$$u_1 \text{ Sin } i = u_2 \text{ Sin } r.$$

$$u_1 \text{ Sin } (d-b) = u_2 \text{Sin } (d-e)$$

For small angles we may omit the sines.

$$u_1 (d-b) = u_2 (d-e)$$

Substituting the tan.

$$u_1 \left(\frac{TY}{YC} - \frac{TY}{YO}\right) = u_2 \left(\frac{TY}{YC} - \frac{TY}{YI}\right)$$

$$u_1 \left(\frac{1}{YC} - \frac{1}{YO}\right) = u_2 \left(\frac{1}{YC} - \frac{1}{YI}\right)$$

For paraxial rays, Y and V are coincident.

$$u_1 \left(\frac{1}{VC} - \frac{1}{VO}\right) = u_2 \left(\frac{1}{VC} - \frac{1}{VI}\right)$$

$$u_1 \left(\frac{1}{r} - \frac{1}{f_1}\right) = u_2 \left(\frac{1}{r} - \frac{1}{f_2}\right)$$

$$\frac{u_1}{r} - \frac{u_1}{f_1} = \frac{u_2}{r} - \frac{u_2}{f_2}$$

$$\frac{u_1}{f_1} + \frac{u_2}{f_2} = \frac{u_2-u_1}{r} \quad * \qquad -\frac{u_1}{f_1} + \frac{u_2}{f_2} = \frac{u_2-u_1}{r} \quad *$$

The above are similar in form but different in signs. Keeping in mind the difference in signs, we may express this formula in one standard form:-

X
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$$\boxed{\frac{u_1}{f_1} + \frac{u_2}{f_2} = \frac{u_2-u_1}{r}}$$

B. Analysis.

When the object is removed to an infinite distance from the surface,

$\frac{u_1}{f_1} = \frac{u_1}{\infty} = 0$ and f_2 becomes a principal focus F_2.

Then:-

$$0 + \frac{u_2}{F_2} = \frac{u_2-u_1}{r}$$

$$\frac{u_2}{F_2} = \frac{u_2-u_1}{r}$$

$$\frac{F_2}{u_2} = \frac{r}{u_2-u_1}$$

$$\boxed{F_2 = \frac{u_2 r}{u_2-u_1}}$$

When the image is formed at an infinite distance from the surface,

$\frac{u_2}{f_2} = \frac{u_2}{\infty} = 0$ and f_1 becomes a principal focus F_1.

Then:-

$$\frac{u_1}{F_1} + 0 = \frac{u_2-u_1}{r}$$

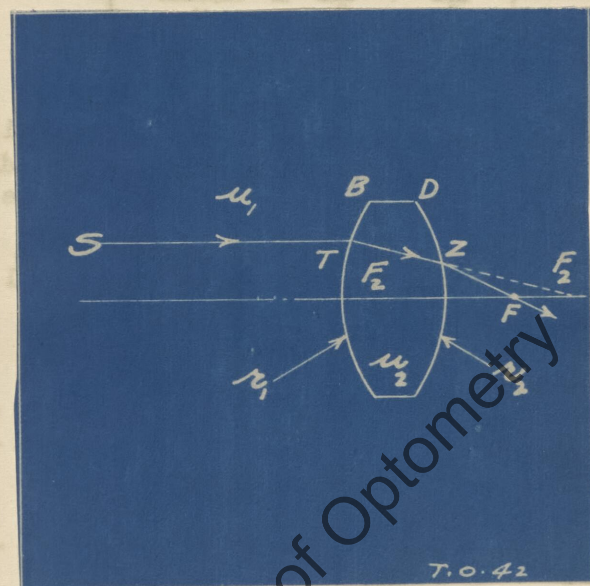
$$\frac{u_1}{F_1} = \frac{u_2-u_1}{r}$$

$$\frac{F_1}{u_1} = \frac{r}{u_2-u_1}$$

$$\boxed{F_1 = \frac{u_1 r}{u_2-u_1}}$$

X
21

* It would appear from the above that the letter r is used to designate two entirely different measurements; the radius of curvature and the angle of refraction. However, since these two never appear in any formula together, no confusion is likely to arise.


S
B
D
T
Z
F
T.O.42

(Note:-
When air is the surrounding medium, the formula becomes:-

$$F_2 = \frac{u_2 r}{u-1}).$$

X

(Note:-
When air is the surrounding medium, the formula becomes:-

$$F_1 = \frac{r}{u-1}).$$

X

IV. Double Refracting Surface (Lens)

A. ST is an incident ray proceeding from a point an infinite distance away from the surface BC.

At T the ray is refracted in the direction TZ and if allowed to proceed would have focussed at F_2. F_2 is therefore the principal focus of the single surface BC.

But the ray TZ is intercepted by the concave surface DK and is refracted again and brought to a final focus at F. F is therefore the principal focus of both surfaces combined; that is of the convex lens.

B. Calculation.

1. Refraction at BC.

$$F_2 = \frac{u_2 r_1}{u_2 - u_1} \quad \text{(vide IV, B).}$$

2. Refraction at DK.

Since DK is concave with respect to the ray TZ we must use a minus sign.

$$\frac{u_1}{F} - \frac{u_2}{F_2} = \frac{u_2 - u_1}{r_2}$$

$$\frac{u_1}{F} - \frac{u_2}{\frac{u_2 r_1}{u_2 - u_1}} = \frac{u_2 - u_1}{r_2}$$

$$\frac{u_1}{F} - \frac{u_2(u_2-u_1)}{u_2 r_1} = \frac{u_2-u_1}{r_2}$$

$$\frac{u_1}{F} - \frac{u_2-u_1}{r_1} = \frac{u_2-u_1}{r_2}$$

$$\frac{u_1}{F} = \frac{u_2-u_1}{r_1} + \frac{u_2-u_1}{r_2}$$

$$\frac{u_1}{F} = (u_2-u_1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$\frac{u_1}{F} = (u_2-u_1)\left(\frac{r_1+r_2}{r_1 r_2}\right)$$

$$\frac{1}{F} = \left(\frac{u_2-u_1}{u_1}\right)\left(\frac{r_1+r_2}{r_1 r_2}\right)$$

$$F = \left(\frac{u_1}{u_2-u_1}\right)\left(\frac{r_1 r_2}{r_1+r_2}\right)$$

$$\boxed{F = \frac{r_1 r_2 u_1}{(r_1+r_2)(u_2-u_1)}}$$

This is the formula for a double refracting surface when immersed in another medium, say a liquid.

(Note:- When the surrounding medium is air, the formula becomes:-

$$F = \frac{r_1 r_2}{(r_1+r_2)(u-1)}$$).

X

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V. Problems.

1. The velocity of light in air is approximately 186,000 miles per second. How fast does it travel in alcohol of index 1.363?
2. Assuming that the velocity of light in air is 30,000,000,000 cms. per second, calculate its velocity in water and in glass.
3. A ray of light is refracted from air into a medium of index $\sqrt{2}$, the angle of incidence being 45°. Find the angle of refraction.
4. Find the angle of incidence of a ray which is refracted at an angle of 30° from air into a medium of index $\sqrt{3}$.
5. Calculate the values of the critical angle for each of the following pairs of media:-
   (a) air and glass;
   (b) air and water;
   (c) air and diamond.
6. Draw diagrams accurately to scale and measure the critical angle in the cases given in problem #5 (glass = 3/2; water = 4/3; diamond = 5/2).
7. A bird is 36 ft. above the surface of a pond. How high does it look to a diver who is under the water? 48′
8. What is the apparent depth of a pool of water 8 ft. deep? 6′
9. Explain how you could determine accurately the index of refraction of a liquid.
10. If an object, viewed normally through a plate of glass with plane parallel faces, seems 5/6 of an inch nearer than it really is, how thick is the glass? 2.5″
11. A layer of ether (u = 1.36) 2 cms. deep, floats on a layer of water (u = 1.33) 3 cms. deep. What is the apparent distance of the bottom of the vessel below the free surface of the ether? 3.73 cm
12. The height of a cylindrical cup is 4 inches and its diameter is 3 inches. A person looking over the rim can just see a point on the opposite side

2.25 inches below the rim. But when the cup is filled with a certain liquid, looking in the same direction as before, he can just see the point of the base farthest from him. Find the index of the liquid. 1.33

13. A fish is 8 ft. below the surface of a pool of clear water. A man shooting at the place where the fish appears to be aims at an angle of 45°. Where will the bullet cross the vertical line that passes through the fish? 3' above fish
14. A pin with a white head is stuck perpendicularly in the center of one side of a flat circular cork, and the cork is floated on water with the pin downwards. Assuming that the head of the pin is 2 inches below the surface of the water, find the smallest diameter the cork can have so that a person looking down through the water could not see the head of the pin. 4.53"
15. Why does the part of a stick obliquely immersed in water appear to be bent up towards the surface of the water? (Illustrate by a diagram.)
16. A cube of glass of index 1.6 is placed on a flat, horizontal picture. Where does the picture appear to be to an eye looking perpendicularly down upon it? 3/8 nearer
17. A ray of light, refracted from air into a parallel plate of transparent refracting material 2 inches thick, makes an angle of 45° with the normal after refraction. The perpendicular displacement of the emergent ray is 3/4 inch. What is the index of refraction of the material?
18. Illustrate graphically the passage of a ray through a prism at:-
    (a) Grazing incidence; (b) Grazing emergence;
    (c) Grazing incidence and emergence.
19. Show that when a ray of light passes through a prism set for minimum deviation, the angle of emergence equals the angle of incidence.
20. The refracting angle of a prism is 60° and the index is $\sqrt{2}$. Show that the angle of minimum deviation is 30°.

21. Find the angle of minimum deviation in the case of a glass prism ($u = 1.54$) of refracting angle 60°.
22. The minimum deviation for a prism of refracting angle 40° is found to be $32^\circ\ 40'$. Find the value of the index of refraction.
23. What must be the refracting angle of a flint-glass prism ($u = 1.6$) to produce the same deviation as is obtained with a crown-glass prism ($u = 1.5$) whose refracting angle is equal to 6°?
24. A glass prism of index 1.5 has a refracting angle of 5°. What is the power of the prism in prism-dioptries?
25. The power of a prism is 2 prism-dioptries and its index is 1.5. Find the refracting angle in degrees.
26. A prism of refracting angle $1^\circ\ 25'$ bends a beam of light through an angle of $1^\circ\ 15'$. Calculate the index of refraction and the power of the prism in prism-dioptries.
27. Combine the following prism corrections into one prism, divide it equally between the two eyes and calculate in what meridian the base-apex lines must be placed.

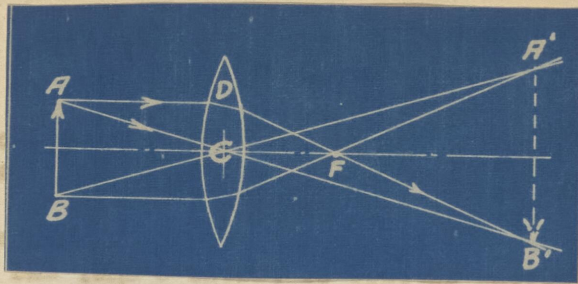
8^Δ base in over O.S.

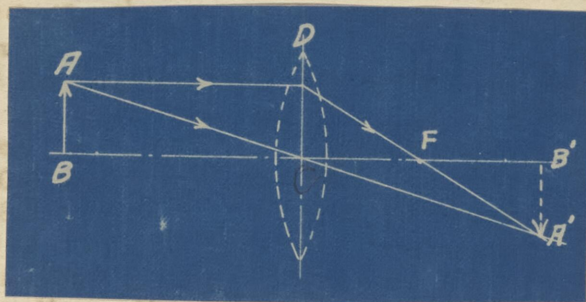
4^Δ base up over O.D.

28. Solve problem #27 by a graphical method.
29. Calculate trigonometrically the resultant of two prisms: one of 4^Δ up and in at 20° and the other of 3^Δ up and in at 50°. Where will the base-apex line of the resultant prism be?
30. Light falling on a concave surface separating water ($u = 1.33$) from glass ($u = 1.55$) is convergent towards a point 10 cms. beyond the vertex. The radius of the surface is 20 cms. Find the point where the refracted rays cross the axis.
31. A small air-bubble in a glass sphere, 4 inches in diameter, viewed so that the speck and the center of the sphere are in line with the eye, appears to

be one inch from the point of the surface nearest the eye. What is its actual distance, assuming that the index of the sphere is 1.5?

32. Assuming that the cornea of the eye is a spherical refracting surface of radius 8 mms. and that the index of the aqueous humor is 4/3, find the distance of the pupil of the eye from the vertex of the cornea, if its apparent distance is found to be 3.04 mms.
33. Calculate the amount of astigmia in diopters, present in a cornea, whose radius in the 180° meridian is 7.5 mms. and in the 90° meridian is 8 mms. ($u = 1.33$).
34. Show that the principal focus of an equi-convex lens of index 1.5 is equal to the radius of either surface.
35. A convex lens has a power of + 5D when in air and of + 1D when immersed in a liquid. What is the index of the liquid, if that of the lens is 1.5?
36. Show that the focal length of a thin plano-convex lens is twice that of an equi-convex lens, if the radii of the curved surfaces are all equal in magnitude.
37. A convex lens has a focal length of 24 inches and an index of 1.5. If the radius of one surface is 6 inches, what is the radius of the other surface?


A
D
A'
C
F
B
B'


D
A
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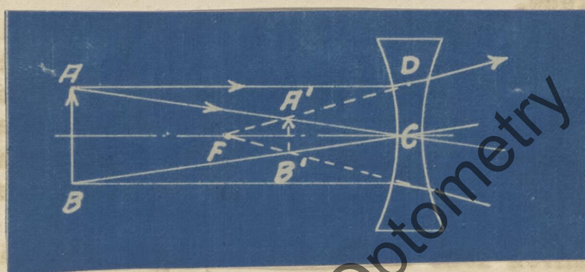
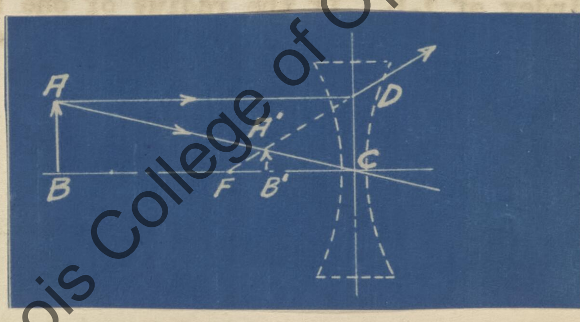
A'


A
D
A'
C
F
B'
B


A
D
A'
C
B
F
B'

CHAPTER III.

LENSES.

I. Thin Lenses.

A. Images (Geometrical)

1. From each extremity of the object draw two lines:
   a. One parallel to the axis which refracts through the P.F.
   b. One through the O.C. which passes unrefracted through the lens.
2. The point where each pair of lines intersects (positively or negatively), is the image of the point from which the lines proceeded.

B. Conjugate Formula.

The triangles ABC and A'B'C are similar.
The triangles DCF and A'B'F are similar.

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C} \;\&\; \frac{DC}{A'B'} = \frac{CF}{B'F}$$

Since AB = DC

$$\frac{BC}{B'C} = \frac{CF}{B'F}$$

$$\frac{f_1}{f_2} = \frac{F}{f_2 - F}$$

$$f_1f_2 - f_1F = f_2F$$

Dividing through by f_1f_2F

The triangles ABC and A'B'C are similar.
The triangles DCF and A'B'F are similar.

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C} \;\&\; \frac{DC}{A'B'} = \frac{CF}{B'F}$$

Since AB = DC

$$\frac{BC}{B'C} = \frac{CF}{B'F}$$

$$\frac{f_1}{f_2} = \frac{F}{F - f_2}$$

$$f_1F - f_1f_2 = f_2F$$

Dividing through by f_1f_2F

$$\frac{f_1 f_2}{f_1 f_2 F} - \frac{f_1 F}{f_1 f_2 F} = \frac{f_2 F}{f_1 f_2 F} \qquad \frac{f_1 F}{f_1 f_2 F} - \frac{f_1 f_2}{f_1 f_2 F} = \frac{f_2 F}{f_1 f_2 F}$$

$$\frac{1}{F} - \frac{1}{f_2} = \frac{1}{f_1} \qquad \frac{1}{f_2} - \frac{1}{F} = \frac{1}{f_1}$$

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} \qquad -\frac{1}{F} = \frac{1}{f_1} - \frac{1}{f_2}$$

These two formulae are similar in form; differing only in signs. Making proper allowance for signs, we may express the two by one standard formula.

$$\boxed{\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}}$$

2 3

(Note:-

This same formula applies to both lenses and mirrors; convex or concave.

The kinds of images formed by a convex lens are similar to those formed by a concave mirror; and those formed by a concave lens to those formed by a convex mirror.)

C. Magnification.

$$\frac{AB}{A'B'} = \frac{BC}{B'C}$$ (diagrams of previous section).

$$\frac{O}{I} = \frac{f_1}{f_2}$$

$$\frac{I}{O} = \frac{f_2}{f_1}$$

24

$$\boxed{M = \frac{SI}{SO} = \frac{DI}{DO}}$$

(Note: The above formula is identical with that for mirrors.)

D. Newton's Formula.

Sir Isaac Newton evolved a formula connecting the conjugate points on the axis of a lens, taking into account the distance of each conjugate from the nearer principal focal point.

This formula is useful when it is more practicable to measure the distance of the object and its image from their respective principal foci, instead of the optical center of the lens itself.

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$\frac{1}{F} = \frac{1}{F + x} + \frac{1}{F + y}$$

$$\frac{1}{F} = \frac{(F + y) + (F + x)}{(F + x)(F + y)}$$

$$(F + x)(F + y) = F(F + y) + F(F + x)$$

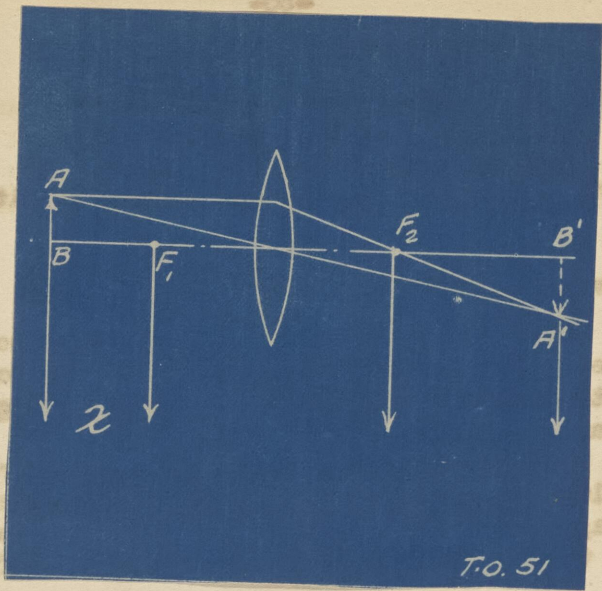
$$F^2 + Fx + Fy + xy = F^2 + Fy + F^2 + Fx$$

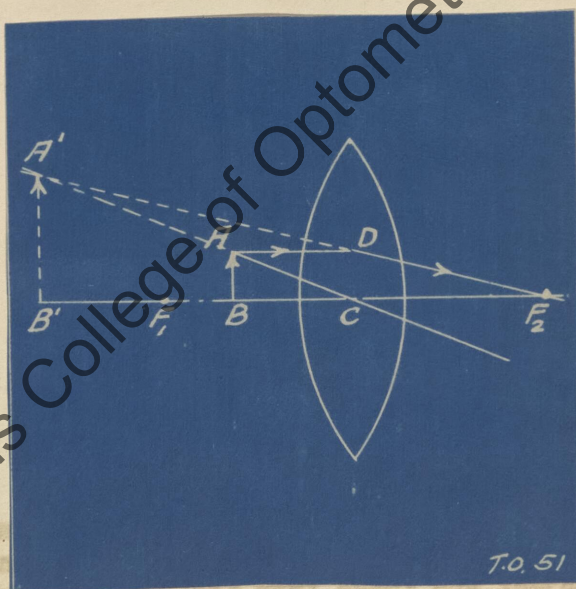
$$xy = F^2$$

25

$$\boxed{F^2 = xy}$$

"The square of the principal focal length of a lens is equal to the product of the distances of the object and image from their respective principal foci."


A
B
F1
F2
B'
A'
x
T.O. 51


A'
A
D
B'
F1
B
C
F2
T.O. 51

E. Magnifying Power.

If a convex lens is held so that an object comes within the principal focus, a virtual image is seen, which is larger than the object. The amount of magnification may be calculated as follows:-

$$\frac{1}{F} = \frac{1}{f_1} - \frac{1}{f_2}$$

$$\frac{1}{F} = \frac{f_2 - f_1}{f_1 \ f_2}$$

By convention, the image is supposed to be found at a distance of 10 inches from the lens.

$$\frac{1}{F} = \frac{10 - f_1}{10 \ f_1}$$

$$10 \ f_1 = 10 \ F - Ff_1$$

$$10 \ f_1 + Ff_1 = 10 \ F$$

$$(10 + F) \ f_1 = 10 \ F$$

$$f_1 = \frac{10 \ F}{10 + F}$$

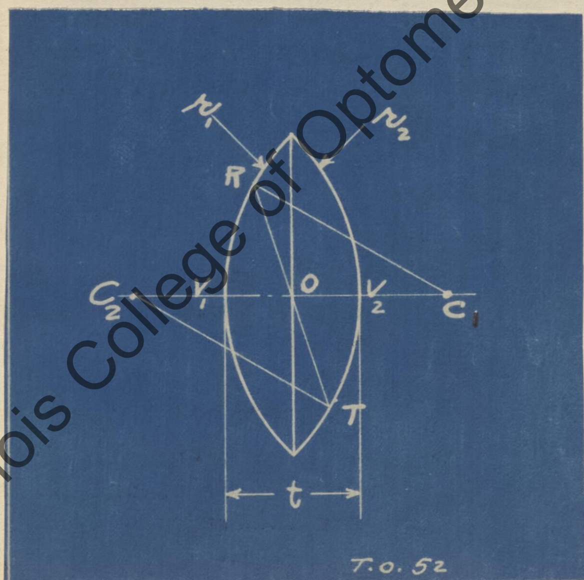
But:

$$M = \frac{f_2}{f_1} = \frac{10}{f_1}$$

$$= \frac{10}{10F/10+F}$$

$$= \frac{10(10+F)}{10F}$$

$$= \frac{10+F}{F} = \frac{10}{F} + \frac{F}{F}$$


r1
r2
R
C2
V1
O
V2
C1
T
t
T.O. 52

$$= 1 + \frac{10}{F}$$

Since $F = \frac{40}{D}$

$$M = 1 + \frac{10}{40/D}$$

$$= 1 + \frac{10\ D}{40} = 1 + \frac{D}{4}$$

$$\boxed{M = 1 + \frac{10}{F}} \quad \text{or} \quad \boxed{M = 1 + \frac{D}{4}}$$

26

(Notes:-

1. If the lens is held a small distance d from the eye, the formula becomes:-

   $M = 1 + \frac{10-d}{F}$.

2. An error of refraction in the observer's eye will also change the magnifying power. With an emmetropic eye, the formula is simplified to

   $M = \frac{10}{F}$).

F. Optical Center.

1. Definition.

   "The optical center of a lens is that point where the line joining two homologous points crosses the axis."

2. Calculation.

   RC_1 and TC_2 are radii of the respective surfaces, and are parallel. The lens thickness is represented by t, and the radii are r_1 and r_2 respectively.

   The triangles RC_1O and TC_2O are similar.

$$\frac{RC_1}{TC_2} = \frac{RO}{TO}$$

Considering paraxial rays only; R and T coincide respectively with V_1 and V_2.

$$\therefore \quad \frac{r_1}{r_2} = \frac{V_1O}{V_2O} = \frac{V_1O}{(t-V_1O)}$$

$$\frac{r_1}{r_2} = \frac{V_1O}{(t-V_1O)}$$

$$r_1t - r_1V_1O = r_2V_1O$$

$$r_1t = (r_1 + r_2)\,V_1O$$

$$\frac{r_1t}{r_1 + r_2} = V_1O. \qquad \text{Similarly, } V_2O = \frac{r_2t}{r_1 + r_2}$$

$$\boxed{\begin{aligned} V_1O &= \frac{r_1}{r_1 + r_2}.t \\ V_2O &= \frac{r_2}{r_1 + r_2}.t \end{aligned}}$$

27

3. Graphical.
   a. Construct the principal section of the lens with its radii and its thickness drawn to a convenient scale.
   b. Draw a radius to each surface, parallel to each other, and join the homologous points.
   c. The ratio that V_1O bears to the thickness as measured by the scale, multiplied by the actual thickness of the lens, will give the position of the O.C. of the given lens.

G. Back Focal Distance (Effectivity).

1. Altered Position.

If a lens of focal length F brings parallel rays to a focus at a certain plane, then advancing the lens a distance d, will advance the focus the same amount.

The same effect is produced if the original lens were replaced in its first position by a lens of focal length (F-d).

The power of the second lens is called the effectivity of the first lens after displacement, when referred to the original plane before displacement. When displaced through a distance d, the effectivity is expressed thus:-

$$\frac{1}{F_B} = \frac{1}{F-d} \text{ (for a convex lens)}$$

$$\frac{1}{F_B} = \frac{1}{-F-d} \text{ (for a concave lens)}$$

(Notes:- After advancement:-

a. A plus lens increases its effectivity.

b. A minus lens decreases its effectivity.)

We may, making due allowance for signs, express the two equations above as one standard equation.

$$\boxed{\frac{1}{F_B} = \frac{1}{F-d}}$$

2 8

2. Change of Effect.

The altered effect of a lens moved from one position to another in front of a plane is the difference between its effectivity in its original and in its new position.

$$\boxed{\frac{1}{F_c} = \frac{1}{F-d_2} - \frac{1}{F-d_1}}$$ 29

3. Two Lenses.

a. Contact.

The combined power of two thin lenses in contact is equal to the sum of their individual powers.

$$\frac{1}{F} = \frac{1}{F_1} + \frac{1}{F_2}$$

$$\frac{1}{F} = \frac{F_2 + F_1}{F_1F_2}$$

$$F = \frac{F_1F_2}{F_2 + F_1}$$

$$\boxed{F = \frac{F_1F_2}{F_1 + F_2}} \quad \text{(contact)}$$ 30

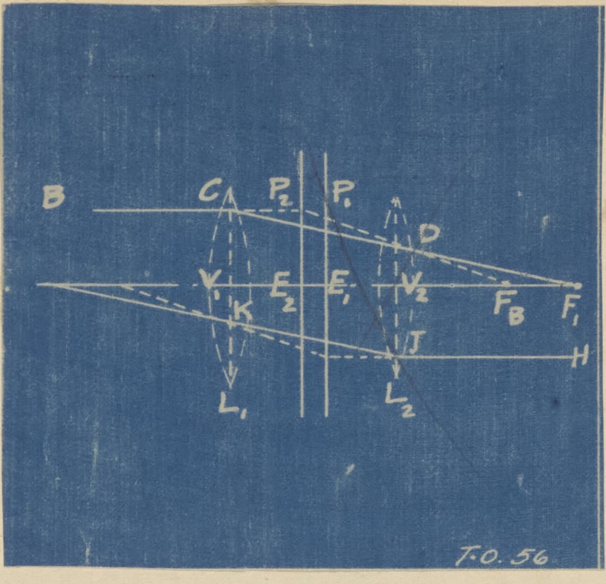
b. Separated.

If the two lenses are separated by a distance d, the resultant effect of the combination is equal to the sum of the power of the one and the effective power of the other at the plane of the ~~first.~~ second

$$\frac{1}{F_B} = \frac{1}{F_1-d} + \frac{1}{F_2}$$

$$\frac{1}{F_B} = \frac{F_2 + (F_1-d)}{(F_1-d)\ (F_2)}$$

$$\frac{1}{F_B} = \frac{F_1 + F_2-d}{(F_1-d)\ F_2}$$


B
C
P2
P1
D
V1
E2
E1
V2
FB
F1
K
J
H
L1
L2
T.O. 56

$$\boxed{F_B = \frac{(F_1-d).F_2}{F_1 + F_2-d}}$$

31

H. Equivalence.

Any number of lenses can always be replaced by a single lens which will give the same amount of magnification as the combination. This single lens is called the equivalent lens.

1. Data

L_1 and L_2 are two thin convex lenses, separated by a distance d. A ray BC, parallel to the common axis, is focused by L_1 towards F_1; but on passing through L_2 is finally focused at F_B.

Produce BC forwards and F_BD backwards until they meet at P_2. P_2E_2 is one of the equivalent planes where the equivalent lens would be situated.

2. Calculation.

$$\frac{E_2F_B}{V_2F_B} = \frac{P_2E_2}{DV_2} = \frac{CV_1}{DV_2} = \frac{V_1F_1}{V_2F_1}$$

$$\frac{E_2F_B}{V_2F_B} = \frac{V_1F_1}{V_2F_1}$$

$$E_2F_B = \frac{V_1F_1 \cdot V_2F_B}{V_2F_1}$$

$$F_E = \frac{F_1 \cdot \dfrac{(F_1-d)F_2}{F_1 + F_2-d}}{(F_1-d)}$$

$$F_E = \frac{F_1F_2(F_1-d)}{F_1 + F_2-d} \times \frac{1}{(F_1-d)}$$

$$\boxed{F_E = \frac{F_1F_2}{F_1 + F_2-d}} \quad \text{or} \quad \boxed{D_E = D_1 + D_2 - d\, D_1D_2}$$ (dioptral formula) 32

The distance of the second equivalent plane from $L_2 = E_2V_2$

$$E_2V_2 = E_2F_B - V_2F_B$$

$$E_2 = F_E - F_B$$

$$E_2 = \frac{F_1F_2}{F_1 + F_2-d} - \frac{(F_1-d)\,F_2}{F_1 + F_2-d}$$

$$E_2 = \frac{F_1F_2 - F_1F_2 + F_2d}{F_1 + F_2-d}$$

$$E_2 = \frac{F_2d}{F_1 + F_2-d}.$$ Similarly $E_1 = \dfrac{F_1d}{F_1 + F_2-d}$.

$$\boxed{\begin{aligned} E_1 &= \frac{F_1d}{F_1 + F_2-d} \\ E_2 &= \frac{F_2d}{F_1 + F_2-d} \end{aligned}}$$ 33

(Note:-

The distance of each equivalent plane, if positive, is measured towards the other lens; and if negative, away from the other lens.)

The equivalent thickness, or optical interval, is expressed by the following equation.

$$\boxed{t = \frac{d^2}{F_1 + F_2 - d}}$$

34

II. Thick Lenses.

A. Focal Length.

A thin lens is one which is regarded as having no appreciable thickness in relation to its focal length; so that the refractions caused by its two surfaces are presumed to have taken place at the plane passing through its optical center. With a thick lens, this assumption is not permissible.

The expression of the focal length of a thick lens is a combination of the formula for a double refracting surface and that for the amount of displacement in a line perpendicular to the faces of a refracting surface.

$$\boxed{F_E = \frac{r_1 r_2}{(r_1 + r_2 - \frac{t(u-1)}{u})(u-1)}}$$

35

B. Vertex Refraction.

1. Vertex Dioptry.

When light from an object reaches the eye, the final position of the image seen depends upon the kinds and the amounts of refractions which have taken place along the path of the light.

In the case of a thin lens, there is supposed to be but one refraction, and that is at the plane of the O.C. of the lens.

A thick lens however, produces two refractions: one at the anterior surface (that nearer the source of light), and the other at the posterior surface (that nearer the eye). The extra distance from the optical center to the posterior pole (vertex) must now be taken into account.

This extra amount of glass produces the effect of adding plus power to a lens as measured in the ordinary dioptry system. In the vertex system therefore, plus lenses have their powers increased, while minus lenses have their powers decreased a certain amount.

The amount of this change is given by the expression $\frac{t}{u} \cdot D_1^2$, where t is the thickness of the lens in meters, u is the refractive index and D_1 is the dioptric value of the surface nearer the light-source.

The new vertex value of a lens compared with its value in the dioptry system is expressed as follows:

$$D = D_1 + D_2 \quad \text{(dioptry system).}$$

$$\boxed{D_v = D_1 + D_2 + \frac{t}{u} \cdot D_1^2}$$ 36

2. Neutralization.

When neutralizing a lens, it is best to place the neutralizing lens in contact with the posterior surface of the lens to be neutralized. In lenses of toric form, this is impracticable; so the neutralizing lens is held in contact with the anterior surface. This changes the vertex formula a little, substituting D_2 for D_1 in the extra portion of the formula.

$$\boxed{D_N = D_1 + D_2 + \frac{t}{u} \cdot D_2^2} \quad \text{(Neutralizing)}$$ 37

3. Alternative Formula.

The use of the vertex formula to ascertain the value of D_1 when all the other values are known, involves the solution of a quadratic. To avoid the use of a quadratic, the following alternative is offered:-

$$\boxed{D_V = \frac{D_1 + D_2 - \frac{t}{u} D_1 D_2}{1 - \frac{t}{u} D_1}}$$ (alternative) 38

III. Problems.

1. An engraver uses a magnifying glass of focal length + 4 inches, holding it close to his eye. At what distance must the lens be from the work so that the magnification may be fourfold?
2. Assuming that the optical system of the eye is equivalent to a thin convex lens of focal length 15 m/ms; what will be the size of the retinal image of a child 3 ft. 4 ins. high at a distance of 15 meters from the eye?
3. The distance between an object and its real image in a thin lens is 32 inches. If the image is 3 times as large as the object, find the position and focal length of the lens.
4. A far-sighted person can see distinctly only at a distance of 40 cms. or more. How much will his range of distinct vision be increased by using spectacles of focal length + 32 cms?
5. An object is to be placed in front of a convex lens of focal length 18 inches, in such a position that its image is magnified 3 times. Find the two possible positions of the object.
6. Light from an object 20 inches in front of two lenses in contact focuses 40 inches behind the combination.

   When one of the lenses is removed, the light focuses 10 inches back of the remaining lens. What was the dioptric power of the lens removed? (-3 D)
7. Light from an object 4 inches outside the anterior principal focus of a convex lens is brought to a focus 16 inches beyond the posterior principal focus. What is the focal length of the lens?
8. What is the magnifying power of a + 20D lens in the following cases:-

(a) A hyperope whose distance of distinct vision is 16 inches.
(b) A myope whose distance of distinct vision is 6 inches.
(c) An emmetrope whose distance of distinct vision is 10 inches.

9. Calculate the position of the O.C. of a double convex lens of radii 25 cms and 5 inches, and whose thickness is 6 m/ms.
10. Calculate the position of the O.C. of a meniscus lens under the following conditions, if its thickness is 4 m/ms.
    (a) Radii + 10" and - 6".
    (b) Radii + 6" and - 10".
11. Solve questions #9 and #10 graphically.
12. What is the power of the single lens which is equivalent to a - 5D lens and a - 4D lens separated by a distance of 2 inches?
13. What is the equivalent single lens of the following conbination;- a + 5D lens and a +2D lens separated by a distance of 25 cms?
14. A - 10D lens is held by a spectacle frame 20 cms. from the eye. What is the effectivity of this lens at a plane 15 m/ms. from the eye?
15. What is the change of effect in moving a 10D lens from a position 15 m/ms. in front of an eye, to a position 20 m/ms. in front of the eye, if:-
    (a) The lens is convex.
    (b) The lens is concave.
16. What is the focal length of a thick biconvex lens of radii 10 cms. and 6 cms. respectively, if its index is 1.5 and its thickness is 3 cms?
17. A + 4D lens is ground on a - 6D base and has an index of 1.52 and a thickness of 4.56 mms. What is its power in the vertex system?
18. What lens from the trial case will exactly neutralize a lens 5.5 m/ms. thick, of index 1.523, ground on a -6D base, with a + 11.5D outside curve?

19. What power must be given to the anterior surface of a lens ground on a - 6D base, so that its vertex refraction is exactly + 5.5D, if its thickness is 5.5 m/ms. and its index is 1.523 (check your result).
20. Solve problem #19, using the alternative formula.

CHAPTER IV.

APPLIED OPTICS.

I. Laps and Lens Measures.

When a grinding tool or a lens measure is made, it is made to be used on glass of a specific index. If, for any reason, either is used on glass of a different index from the specified one, the power of the curve ground, or the one measured, will not be the true power, as indicated by the lap or lens measure. To determine the true power, a calculation must be made.

The radius produced is constant, let us say r.

Let D_R be the real dioptric power produced or measured by either agent on glass of u_R.

Let D_A be the apparent power produced when used on glass of index u_A.

From the formula of a single refracting surface:-

$$D_R = \frac{u_R - 1}{r} \quad \text{and} \quad D_A = \frac{u_A - 1}{r}$$

$$r = \frac{u_R - 1}{D_R} \quad \text{and} \quad r = \frac{u_A - 1}{D_A}$$

Hence:-

$$\frac{u_A - 1}{D_A} = \frac{u_R - 1}{D_R}$$

$$\boxed{\frac{D_R}{D_A} = \frac{u_R - 1}{u_A - 1}}$$

39

"The ratio of the real dioptric power to the apparent dioptric power is equal to the ratio of the decimal part of the real index to the decimal part of the apparent index."

II. Powers in Oblique Cylindrical Meridians.

A cylinder has two principal meridians: one in which there is no power (axis) and the other in which the power is at a maximum. Between these two meridians, the power varies from meridian to meridian.

The power in any meridian (D_M), of any cylinder (D_c), when that meridian makes an angle (a) with the axis of the cylinder, is calculated from:-

$$\boxed{D_M = D_c \operatorname{Sin}^2 a.}$$ 40

III. Tilted Sphere.

When a spherical lens is tilted forward before an eye, this tilting introduces slight cylindrical power into the lens.

The amount of cylinder developed (D_c) depends upon the power of the sphere (D_s) and the angle of tilt (b).

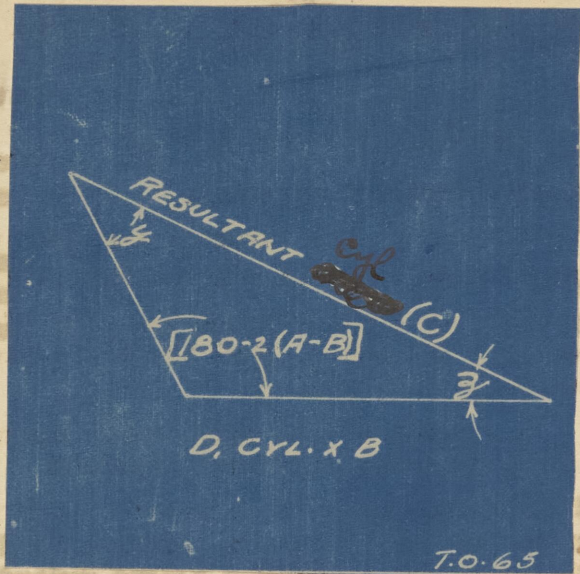
When tilted through small angles and when small spherical powers are in question, the amount of cylindrical power introduced by the tilting is calculated from the following formula:-

$$\boxed{D_c = D_s \tan^2 b.}$$ 41

IV. Obliquely Crossed Cylinders.

Optometrists sometimes find it necessary to prescribe cylinders whose axes are <u>not</u> at right angles to each other. It is not an easy matter to grind a lens in this form. It is better to find the equivalent sphero-cylindrical compound which will have exactly the same optical effect as the obliquely crossed cylinders.

The problem is to transpose:-


RESULTANT
Cyl
[redacted]
(C)
y
[180-2(A-B)]
z
D, CYL. X B
T.O.65

D_1 cyl. ax b $\frown$ D_2 cyl. ax. (b + a)

into

S sph. $\frown$ C cyl. ax (b + x).

A. Trigonometrical Solution.

D_1 denotes the power of the oblique cylinder whose axis-slope is the smaller of the two.

a = the angle between the axes of the two oblique cylinders, that is, (axis slope of D_1 - axis slope of D_2).

C = power of the cylindrical element of the final sphero-cylindrical compound.

x = the angle to be added to the smaller axis slope of the oblique cylinders, to give the axis slope of the cylinder in the final sphero-cylinder compound.

S = the power of the sphere in the sphero-cylinder compound.

$$C = \sqrt{D_1^2 + D_2^2 + 2\,D_1 D_2 \text{Cos. } 2a}$$

$$S = \frac{D_1 + D_2 - C}{2}$$

$$\tan 2x = \frac{D_2 \text{ Sin } 2a}{D_1 + D_2 \text{Cos } 2a}$$

42

B. Graphical Solution:

1. Power of the Resultant Cylinder.
   a. Transpose so that both cylinders have the same sign (preferably +).
   b. Find the angle between the axes of the two oblique cylinders, double it, and subtract the doubled angle from 180°.
   c. Draw two straight lines containing the angle given in (b).

d. Using a convenient scale, mark the power of one oblique cylinder on one line, and the power of the other oblique cylinder on the other line.

(Note:- The lines will not be parallel to the axes of the oblique cylinders)

e. Complete the triangle. This third side gives the power of the resultant cylinder.

2. Axis of the Resultant Cylinder.

a. Measure, with a protractor, the angles that this third side makes with each of the other two cylinders, and divide each by 2.

(Note:- The sum of all three angles of the triangle must equal 180°).

b. The axis of the resultant cylinder is the number of degrees away from each oblique cylinder axis, as given by the division under (a); the resultant angle, by addition or subtraction, being the same in each case.

(Note the angle and the axis from which it is to be measured, are adjacent to each other.)

3. Power of the Sphere.

a. Add the powers of the oblique cylinders, subtract the power of the resultant cylinder from their sum, and divide by 2.

(Note:- This gives the sphere resulting from the combination of the two obliquely crossed cylinders.)

b. Combine the sphere given by (a) with the original sphere of the transposed Rx (if there is one), and this gives the resultant sphere of the final sphero-cylindrical compound.

4. Sphero-Cylinder Compound.

a. 1,e. above gives the power of the resultant cylinder.

b. 2,b. above gives the axis of the resultant cylinder.

c. 3,b. above gives the power of the resultant sphere.

d. Combination of the results above will give the final R_x.

V. Lens, Object and Image.

An interesting problem in applied optics is to find the position of a lens for a given distance between an object, and its image formed by that lens. The solution involves a quadratic.

The general form of the final equation is evolved as follows:-

d = distance between object and its image.
f_1 = distance of object from the lens.
$(d-f_1)$ = distance of image from the lens.
F = focal length of the lens.

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{d-f_1}$$

$$\frac{1}{F} = \frac{d-f_1 + f_1}{f_1(d-f_1)}$$

$$\frac{1}{F} = \frac{d}{df_1 - f_1^2}$$

$$df_1 - f_1^2 = dF$$

$$- f_1^2 + df_1 - dF = 0$$

$$\boxed{f_1^2 - df_1 + dF = 0}$$

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This is a quadratic in f_1 and since d and F are known, we can determine the value of f_1, which will give us the position of the lens.

$$f_1 = \frac{-(-d) \pm \sqrt{(-d)^2 - (4x1xdF)}}{2 \times 1}$$

$$f_1 = \frac{d \pm \sqrt{d^2 - 4dF}}{2}$$

IV. Problems.

1. A surface grinder is presented with a piece of glass of index 1.6 and is asked to grind a + 12D curve on one side. The only laps available are those made for glass of index 1.5. What lap must he use to produce the required curve?
2. A meniscus lens of index 1.6 has powers of + 12D and - 6D when measured with a lens measure. But the lens measure is calibrated for glass of index 1.5. What is the actual power of the surfaces?
3. What is the power in each of the given meridians of a + 5D cylinder:-

   (a) 15°; (b) 30°; (c) 45°; (d) 60°; and (e) 75°.

4. What is the amount of the cylindrical effect produced by tilting the given lenses through 5° and 10°:-

   (a) + 2D; (b) + 5D; (c) + 2 $\subset$ + 3 x 180°; and (d) + 2 $\subset$ + 2 x 90°.

5. Determine the equivalent sphero-cylindrical combination of each of the following:-

   (a) + 4.00 ax 20° $\subset$ - 2.75 ax 65°

   (b) - 2.50 ax 30° $\subset$ + 1.75 ax 90°

   (c) + 3.00 ax 70° $\subset$ + 2.00 ax 20°

   (d) + 4.00 ax 10° $\subset$ + 4.00 ax 60°

   (e) + 4.00 ax 60° $\subset$ - 4.00 ax 120°

   (f) - 1.75 ax 120° $\subset$ + 1.25 ax 135°

(g) + 2.25 ax 40° ⊃ - 4.00 ax 115°

6. Solve each of the problems in question #5, by a graphical method.
7. Where would you place the following lenses so that the distance between the object and its image shall be as indicated:-
   (a) 7 inch lens with object and image 36 inches apart.
   (b) + 8D lens separating object and image by 21.8 inches.
   (c) - 10D lens with 13.33 inches between object and image.

Finis.

	Mon	Tues	Wed	Thurs	Fri
1	Theor optics	Theor optics Sec 3	Optical Math	Theor. optics	Physiology
2.	Ocular anatomy	Ocular anatomy	Theor optics	Optometry	Rx
3.	Optom. Section 2	Physiology	Opt. Forbes	Physiology	Opt math
4.	Opt Section 1	Optometry Sec 2	Ocular Anat Sec 3	Optom Pokie	Ocular anat.
5.	Ocular anat Miss W.	Theor optics	Theor optics	Ocular anat Walker	Theor optics